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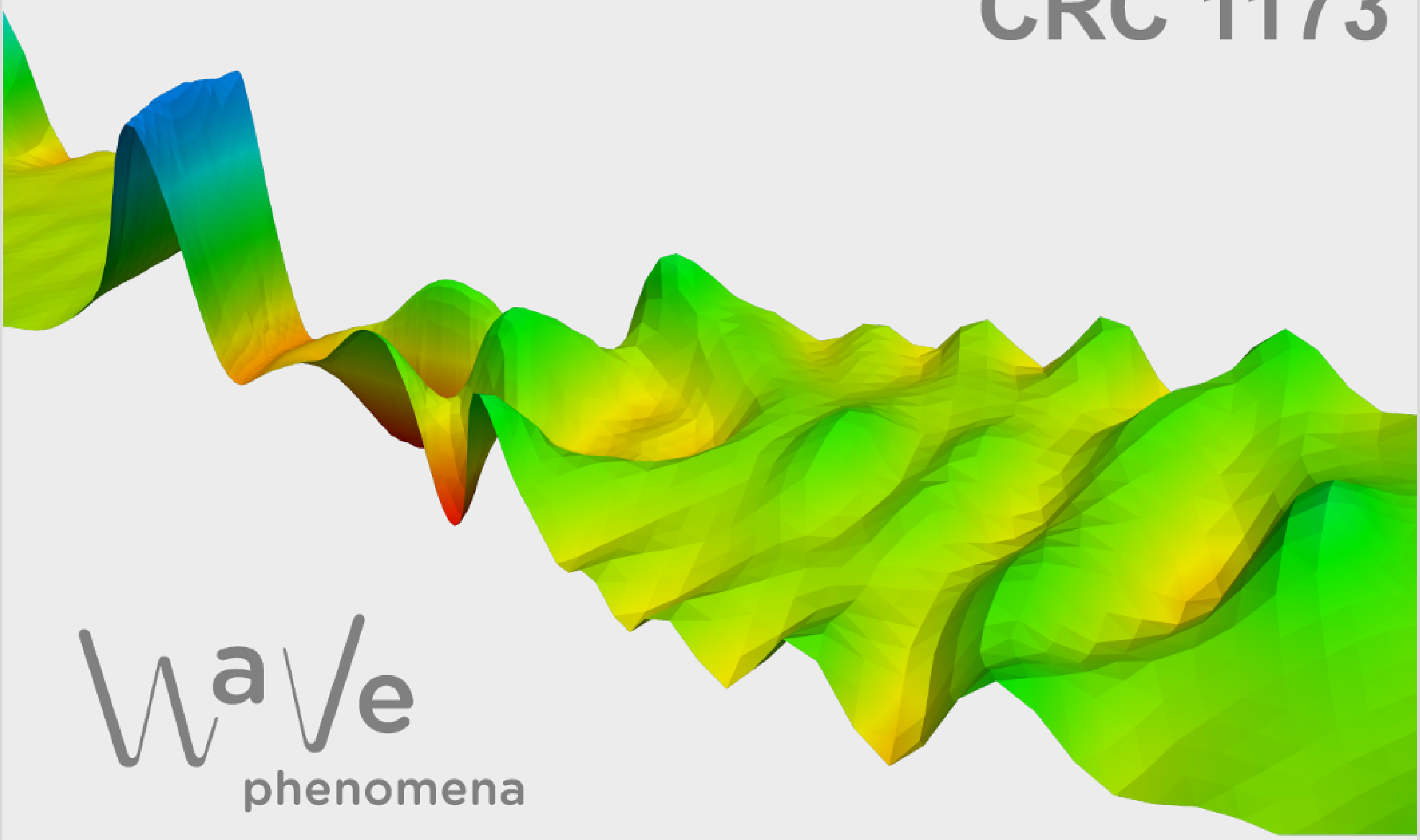

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BIHARMONIC WAVE MAPS: LOCAL WELLPOSEDNESS IN HIGH REGULARITY

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Abstract. We show a local wellposedness result for biharmonic wave maps with initial data of sufficiently high Sobolev regularity. Moreover, we obtain a blow-up criterion for these solutions. In contrast to the wave maps equation we use a vanishing viscosity argument and an appropriate parabolic regularization in order to obtain the existence result. The geometric nature of the equation is exploited to prove convergence of the approximate solutions and uniqueness of the limit.

1. Introduction

Let (N, g) be a smooth and compact Riemannian manifold which we assume to be isometrically embedded into some Euclidean space $\mathbb{R}^L$. Biharmonic wave maps are critical points $u : \mathbb{R}^n \times [0, T) \to N$ of the (extrinsic) action functional

$$\Phi(u) = \frac{1}{2}\int_0^T \int_{\mathbb{R}^n} |\partial_t u|^2 - |\Delta u|^2 \; dx \; ds. \tag{1.1}$$

The Euler-Lagrange equation has been calculated in [HLS18] (in the case $N = S^l \subset \mathbb{R}^{l+1}$) and in [Sch18] (for arbitrary N) and it is given by

$$\partial_t^2 u + \Delta^2 u \perp T_u N, \quad \text{on } \mathbb{R}^n \times [0, T). \tag{1.2}$$

In order to obtain a more explicit form of this equation we use the fact that there exists some $\delta_0 > 0$ and a smooth family of linear maps $P_p : \mathbb{R}^L \to \mathbb{R}^L$ for $\operatorname{dist}(p, N) < \delta_0$, such that

$$P_p : \mathbb{R}^L \to T_p N, \; p \in N$$

is an orthogonal projection onto the tangent space $T_p N$. Thus, the Euler-Lagrange equation (1.2) can be written as

$$\partial_t^2 u + \Delta^2 u = (I - P_u)(\partial_t^2 u + \Delta^2 u).$$

Using the fact that u takes values in N we calculate

$$\begin{aligned} \partial_t^2 u + \Delta^2 u =& dP_u(u_t, u_t) + dP_u(\Delta u, \Delta u) + 4dP_u(\nabla u, \nabla \Delta u) + 2dP_u(\nabla^2 u, \nabla^2 u) \\ &+ 2d^2 P_u(\nabla u, \nabla u, \Delta u) + 4d^2 P_u(\nabla u, \nabla u, \nabla^2 u) \\ &+ d^3 P_u(\nabla u, \nabla u, \nabla u, \nabla u) \\ =:& \mathcal{N}(u). \end{aligned} \tag{1.3}$$

The the main goal of this paper is to show the following local wellposedness result for the Cauchy problem for (1.2) in Sobolev spaces with sufficiently high regularity.

Theorem 1.1. *Let $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}^L$, $u_0(x) \in N$, $u_1(x) \in T_{u_0(x)} N$, for a.e. $x \in \mathbb{R}^n$ and such that*

$$(\nabla u_0, u_1) \in H^{k-1}(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n)$$

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for some $k \in \mathbb{N}$ with $k > \lfloor \frac{n}{2} \rfloor + 2$. Then there exists $T = T(\|\nabla u_0\|_{H^{k-1}}, \|u_1\|_{H^k}) > 0$ and a unique solution $u : \mathbb{R}^n \times [0,T) \to N$ of (1.2) with

$$u - u_0 \in L^\infty([0,T), H^k(\mathbb{R}^n)) \cap W^{1,\infty}([0,T), H^{k-2}(\mathbb{R}^n)),$$

which is weakly continuous, i.e., $(u, \partial_t u)$ is weakly continuous in $H^k \times H^{k-2}(\mathbb{R}^n)$. Further, the solution u extends beyond T if

$$\int_0^T \|\nabla u(s)\|_{W^{1,\infty}}^{2k} + \|u_t(s)\|_{L^\infty}^{2k} \; ds < \infty. \tag{1.4}$$

It is worthwhile to remark that both u_0 and $u(t)$ do not necessarily belong to $L^2(\mathbb{R}^n)$ and it is only the difference of these two functions which belongs to this space.

The first, second and fourth author have recently shown in [HLS18] that there exists a global weak solution of (1.2) for initial data in the energy space $H^2 \times L^2$ in the case $N = S^l \subset \mathbb{R}^{l+1}$. In [HLS18] a crucial ingredient is a conservation law which allows to obtain the desired solution as a weak limit of a sequence of solutions of suitably regularized problems. The derivation of this conservation law relies on the fact that the action functional Φ is invariant under rotations in the highly symmetric setting $N = S^l$ and this argument does not apply to arbitrary target manifolds N.

Moreover, the third author has shown energy estimates for biharmonic wave maps in low dimensions $n = 1, 2$ in [Sch18]. When combining this result with Theorem 1.1 , more precisely the blow-up criterion (1.4), he then obtained the existence of a unique global smooth solution of (1.2) for smooth and compactly supported initial data. This results extends earlier work of Fan and Ozawa [FO10] in which they only considered spherical target manifolds.

A local well-posedness result as in Theorem 1.1 is standard for second-order wave equations such as wave maps and it can be found for example in the books of Shatah and Struwe [SS98] and Sogge [Sog08]. Here the difference is that the nonlinearity $\mathcal{N}(u)$ depends on the third spatial derivative of u whereas the energy only contains second spatial derivatives and in our proof we use the geometric nature of the equation in several crucial steps in order to be able to rewrite this expression in terms of derivatives of lower order. This makes the argument fairly delicate.

In the following we briefly outline the structure of the paper. Since the nonlinearity $\mathcal{N}(u)$ in equation (1.3) contains derivatives of up to third order we cannot directly apply the energy estimates for the operator $\partial_t^2 + \Delta^2$ and construct the desired solution by means of a fixed-point argument. Instead, in Section 3, we use a vanishing viscosity approximation and solve the corresponding Cauchy problem for the damped plate operator

$$\partial_t^2 u + \Delta^2 u - \varepsilon \Delta \partial_t u \perp T_u N, \quad \varepsilon \in (0,1].$$

In order to obtain a limiting solution for (1.2) as $\varepsilon \searrow 0$, we prove a priori energy estimates which are uniform in ε in Section 4. As a byproduct we obtain the blow up criterion in Theorem 1.1. The existence part of Theorem 1.1 is then shown in Section 5 and in Section 6 we prove that the solutions are unique.

2. Notation and preliminaries

We note that the projector maps P_p defined in the introduction are derivatives of the metric distance (with respect to N) in $\mathbb{R}^L$, i.e.,

$$p = \pi(p) + \frac{1}{2}\nabla_p(\mathrm{dist}^2(p,N)), \quad P_p = d_p\pi(p), \quad \mathrm{dist}(p,N) < \delta_0. \tag{2.1}$$

Moreover, if $p \in \mathbb{R}^L$ is sufficiently close to N, then π has the nearest point property, i.e., $|\pi(p) - p| = \inf_{q \in N} |q - p|$, and thus

$$d\pi_{|_p} = d\pi(p) = d(\pi^2(p)) = d\pi_{|_{\pi(p)}} d\pi_{|_p}.$$

Thus $P_p : \mathbb{R}^L \to T_{\pi(p)}N$ is well-defined. Using cut-off functions we extend the identity (2.1), and thus also the equation $P_p = d_p\pi(p)$, to all of $\mathbb{R}^L$. This shows that when one tries to solve (1.3) one does not have to restrict the coefficients a priori.

In the following we use the shorthand $\nabla^{k_1}u \star \nabla^{k_2}u$ for (linear combinations of) products of partial derivatives of u of order $k_1 \in \mathbb{N}$ and $k_2 \in \mathbb{N}$. With this notation we can rewrite equation (1.3) as

$$\begin{aligned}\partial_t^2 u + \Delta^2 u = dP_u(u_t u_t + \nabla^2 u \star \nabla^2 u + \nabla^3 u \star \nabla u)\\ + d^2P_u(\nabla u \star \nabla u \star \nabla^2 u) + d^3P_u(\nabla u \star \nabla u \star \nabla u \star \nabla u).\end{aligned}$$

Further, for $l \in \mathbb{N}_0$ we denote by d^lP_p the derivative of order l of the map P_p, which is a $(l+1)$-linear form on $\mathbb{R}^L$. The Leibniz formula implies the following Lemma

Lemma 2.1. *For $m \in \mathbb{N},\ l \in \mathbb{N}_0$ we have*

$$\nabla^m(d^lP_u) = \sum_{j=1}^{m} \sum_{\sum_{k=1}^{j} m_k = m-j} d^{j+l}P_u(\nabla^{m_1+1}u \star \cdots \star \nabla^{m_j+1}u), \tag{2.2}$$

According to the remark above,

$$d^{j+l}P_u(\nabla^{m_1+1}u \star \cdots \star \nabla^{m_j+1}u)$$

is a $l+1$ linear form and in order to include the case $m = 0$, we will use $\sum_{j=\min\{1,m\}}^{m}$ for the sum in the formula above.

We include the calculation of derivatives $\nabla^m(\mathcal{N}(u))$ and $\nabla^m(\mathcal{N}(u) - \mathcal{N}(v))$ for sufficiently regular $u, v : \mathbb{R}^n \times [0,T) \to \mathbb{R}^L$ and $m \in \mathbb{N}_0$ using the $\star$-convention in Appendix A. The results from Appendix A will be used frequently throughout the paper.

In the following sections, we also need a version of the classical Moser estimate, see e.g., [Tay11, chapter 13].

Lemma 2.2. *Let $l, k \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_l \in \mathbb{N}_0^n$, $\sum_{i=1}^{l} |\alpha_i| = k$. There exists $C > 0$ such that for all $f_1, \dots, f_l \in C_0(\mathbb{R}^n) \cap H^k(\mathbb{R}^n)$*

$$\|D^{\alpha_1} f_1 \cdot \dots \cdot D^{\alpha_l} f_l\|_{L^2} \le C \prod_{i=1}^{l} \|f_i\|_{L^\infty}^{1-\frac{|\alpha_i|}{k}} \|f_i\|_{H^k}^{\frac{|\alpha_i|}{k}}. \tag{2.3}$$

In particular,

$$\|D^{\alpha_1} f_1 \cdot \dots \cdot D^{\alpha_l} f_l\|_{L^2} \le C \sum_{j=1}^{l} \prod_{i \neq j}^{l} \|f_i\|_{L^\infty} \left(\|f_1\|_{H^k} + \dots + \|f_l\|_{H^k}\right). \tag{2.4}$$

3. Existence for the parabolic approximation

Since

$$\mathcal{N}(u) = \mathcal{N}(u, u_t, \nabla u, \nabla^2 u, \nabla^3 u),$$

energy estimates for the operator $\partial_t^2 + \Delta^2$ are not sufficient to show the existence of a solution of (1.3). Instead, we use the damped plate operator

$$\partial_t^2 + \Delta^2 - \varepsilon\Delta\partial_t,$$

with $\varepsilon \in (0,1]$ fixed, as a regularization. More precisely, we prove the existence of a solution $u^\varepsilon : \mathbb{R}^n \times [0, T_\varepsilon) \to N$ of the Cauchy problem

$$\begin{cases} \partial_t^2 u^\varepsilon(x,t) + \Delta^2 u^\varepsilon(x,t) - \varepsilon\Delta\partial_t u^\varepsilon(x,t) \perp T_{u^\varepsilon(x,t)}N, & (x,t) \in \mathbb{R}^n \times [0, T_\varepsilon),\\ u^\varepsilon(x,0) = u_0(x),\ u_t^\varepsilon(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

where $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}^L$, $u_0(x) \in N$ and $u_1(x) \in T_{u_0(x)}N$ for a.e. $x \in \mathbb{R}^n$, and such that

$$(\nabla u_0, u_1) \in H^{k-1}(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n)$$

for some $k \in \mathbb{N}$ with $k > \lfloor \frac{n}{2} \rfloor + 2$. In the following we drop the super-/subscript ε and write (u, T) instead of $(u^\varepsilon, T_\varepsilon)$. We note that the condition in (3.1) reads as

$$\partial_t^2 u + \Delta^2 u - \varepsilon \Delta \partial_t u = \mathcal{N}(u) - \varepsilon (I - P_u)(\Delta \partial_t u). \tag{3.2}$$

Via the expansion

$$\varepsilon (I - P_u)(\Delta \partial_t u) = \varepsilon d^2 P_u(u_t, \nabla u, \nabla u) + \varepsilon 2 dP_u(\nabla u_t, \nabla u) + \varepsilon dP_u(u_t, \Delta u)$$

we obtain

$$\begin{aligned} \partial_t^2 u + \Delta^2 u - \varepsilon \Delta \partial_t u &= \mathcal{N}(u) - \varepsilon d^2 P_u(u_t, \nabla u, \nabla u) - \varepsilon 2 dP_u(\nabla u_t, \nabla u) - \varepsilon dP_u(u_t, \Delta u) \\ &=: \mathcal{N}_\varepsilon(u). \end{aligned} \tag{3.3}$$

We next solve (3.3) and we recall that only $u(t) - u_0 \in L^2(\mathbb{R}^n)$.

Lemma 3.1. *Let $\varepsilon \in (0,1)$, $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}^L$ with $u_0(x) \in N$ and $u_1(x) \in T_{u_0(x)} N$ for a.e. $x \in \mathbb{R}^n$, and such that*

$$(\nabla u_0, u_1) \in H^{k-1}(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n)$$

for some $k \in \mathbb{N}$ with $k > \lfloor \frac{n}{2} \rfloor + 2$. Then (3.3) has a unique local solution $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}^L$ with

$$u - u_0 \in C^0([0,T), H^k(\mathbb{R}^n)) \cap C^1([0,T), H^{k-2}(\mathbb{R}^n)) \cap H^1([0,T), H^{k-1}(\mathbb{R}^n)) \tag{3.4}$$

and initial data $u(0) = u_0$ and $u_t(0) = u_1$. In addition,

$$\nabla u \in L^2([0,T), H^k(\mathbb{R}^n)) \tag{3.5}$$

and for $0 \le t \le T$

$$\begin{aligned} &\left\|\nabla^{k-2} u_t(t)\right\|_{L^2}^2 + \left\|\nabla^k u(t)\right\|_{L^2}^2 + \varepsilon \int_0^T \left\|\nabla^{k-1} u_t(s)\right\|_{L^2}^2 \, ds + \varepsilon \int_0^T \left\|\nabla^{k+1} u(s)\right\|_{L^2}^2 \, ds \\ &\quad \le C \Big(\int_0^t \int_{\mathbb{R}^n} \nabla^{k-2}(\mathcal{N}_\varepsilon(u)) \cdot \nabla^{k-2} u_t \, dx \, ds + \|\nabla u_0\|_{H^{k-1}}^2 + \|u_1\|_{H^{k-2}}^2 \Big). \end{aligned} \tag{3.6}$$

Before we prove Lemma 3.1, we set $v(x,t) = u(x,t) - u_0(x,t)$ and rewrite (3.3) into

$$\partial_t U + \mathcal{A}_k U = \begin{pmatrix} 0 \\ f_\varepsilon(U) \end{pmatrix}, \ U(0) = \begin{pmatrix} 0 \\ u_1 \end{pmatrix}, \tag{3.7}$$

where $U = \begin{pmatrix} v \\ v_t \end{pmatrix}$ and $f_\varepsilon(U)$ is defined through

$$\begin{aligned} f_\varepsilon(U) : &= \mathcal{N}(v + u_0) - \varepsilon d^2 P_{v+u_0}(v_t, \nabla(v+u_0), \nabla(v+u_0)) \\ &\quad - \varepsilon 2 dP_{v+u_0}(\nabla v_t, \nabla(v+u_0)) - \varepsilon dP_{v+u_0}(v_t, \Delta(v+u_0)) - \Delta^2 u_0. \end{aligned} \tag{3.8}$$

Further the operator $\mathcal{A}_k : H^k(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n) \supseteq \mathcal{D}(\mathcal{A}) \to H^k(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n)$ is defined through

$$\mathcal{A}_k = \begin{pmatrix} 0 & -I \\ \Delta^2 & -\varepsilon \Delta \end{pmatrix}, \ \mathcal{D}(\mathcal{A}) = H^{k+2}(\mathbb{R}^n) \times H^k(\mathbb{R}^n). \tag{3.9}$$

Since the operators $\mathcal{A}_k$, $k \ge 3$, extend each other we drop the subscript k. It is well known that $\mathcal{A}$ is the generator of a analytic C^0-semigroup $\{T_\varepsilon(t)\}_{t \ge 0}$. In fact, in [DS15, Prop. 2.3], it is proven in the case $k = 2$ that $\mathcal{A}$ generates a (unbounded) analytic C^0-semigroup. We record the following known result, see e.g. [LT00, Prop. 0.1] and [Lun18, Prop. 1.13].

Lemma 3.2. *Let $r \in \mathbb{N}_0$, $u_1 \in H^{r+1}(\mathbb{R}^n)$, and $g \in C^0(0,T; H^r(\mathbb{R}^n))$. Then there exists a solution U of the linear equation*

$$\partial_t U + \mathcal{A} U = \begin{pmatrix} 0 \\ g \end{pmatrix}, \ U(0) = \begin{pmatrix} 0 \\ u_1 \end{pmatrix}, \tag{3.10}$$

with

(3.11)
$U \in L^2(0,T;H^{r+4} \times H^{r+2}(\mathbb{R}^n)) \cap C^0(0,T;H^{r+3} \times H^{r+1}(\mathbb{R}^n)) \cap H^1(0,T;H^{r+2} \times H^r(\mathbb{R}^n))$.

We remark that in general the mild solution of (3.10) is given by

$$U(t) = T_\varepsilon(t)\begin{pmatrix}0\\ u_1\end{pmatrix} + \int_0^t T_\varepsilon(t-s)\begin{pmatrix}0\\ g(s)\end{pmatrix}\, ds. \tag{3.12}$$

We have to apply the following energy estimates.

Lemma 3.3. *Let $r \in \mathbb{N}_0$, $g \in C^0(0,T;H^r(\mathbb{R}^n))$, $u_1 \in H^{r+1}(\mathbb{R}^n)$ and $u_0 : \mathbb{R}^n \to \mathbb{R}^L$ with $\nabla u_0 \in H^{r+3}(\mathbb{R}^n)$. Then v from Lemma 3.2 satisfies for $0 \le t \le T < \infty$*

(3.13)
$$\begin{aligned}&\|v_t(t)\|^2_{H^{r+1}} + \|v(t)\|^2_{H^{r+3}} + \frac{\varepsilon}{2}\int_0^T \|\nabla v_t(s)\|^2_{H^{r+1}}\, ds + \frac{\varepsilon}{2}\int_0^T \|\nabla(v+u_0)(s)\|^2_{H^{r+3}}\, ds\\ &\le C(1+T)\left(\varepsilon^{-1}\int_0^T \left\|g(s) + \Delta^2 u_0\right\|^2_{H^r}\, ds + \|u_1\|^2_{H^{r+1}} + \|\nabla u_0\|^2_{H^{r+2}}\right),\end{aligned}$$

and

$$\begin{aligned}&\left\|\nabla^{r+1} v_t(t)\right\|^2_{L^2} + \left\|\nabla^{r+3} v(t)\right\|^2_{L^2} + \varepsilon\int_0^T \left\|\nabla^{r+2} v_t(s)\right\|^2_{L^2}\, ds\\ &\le C\Big(-\int_0^t\int_{\mathbb{R}^n} \nabla^r\left(g(s) + \Delta^2 u_0\right)\cdot\nabla^r\Delta v_t\, dx\, ds + \|u_1\|^2_{H^{r+1}} + \|\nabla u_0\|^2_{H^{r+2}}\Big).\end{aligned} \tag{3.14}$$

Proof. We note that $u = v + u_0$ satisfies

$$\partial_t^2 u + \Delta^2 u - \varepsilon\Delta\partial_t u = g + \Delta^2 u_0 \tag{3.15}$$

in $L^2(0,T;H^r(\mathbb{R}^n))$. We obtain (3.13) from Lemma 3.2 by differentiating (3.15) of order ∇^l and testing with $-\nabla^l\Delta u_t \in L^2_{t,x}$ where $l \in \{0,\dots,r\}$, and

$$\begin{aligned}\frac{d}{dt}\left\|\nabla^{l+1}u_t(t)\right\|^2_{L^2} + \frac{d}{dt}\left\|\nabla^{l+3}u(t)\right\|^2_{L^2} + \varepsilon\left\|\nabla^{l+2}u_t(t)\right\|^2_{L^2}&\\ \le C\varepsilon^{-1}\left\|\nabla^l(g + \Delta^2 u_0)\right\|^2_{L^2} + \frac{\varepsilon}{2}\left\|\nabla^{l+2}u_t(t)\right\|^2_{L^2}&,\end{aligned} \tag{3.16}$$

which makes sense pointwise a.e. Then we integrate this inequality from $t_0 = 0$ to $t \le T$ and further differentiate (3.15) of order ∇^l and test by $\nabla^l\Delta^2 u$. Then we have to integrate by parts (in x), integrate over t as above from $t_0 = 0$ to $t \le T$ and integrate the term

$$\int_0^T\int_{\mathbb{R}^n} \nabla^l\partial_t^2 u\cdot\nabla^l\Delta^2 u\, dx\, ds$$

again by parts (in t and x). It remains to estimate the L^2-norm of $v_t(t)$ and the H^2-norm of $v(t)$. But this follows from testing the equation with u_t and by using the fact that

$$\|u - u_0\|_{L^\infty_t L^2} \le T\,\|u_t\|_{L^\infty_t L^2}.$$

□

Proof of Lemma 3.1. We aim at constructing a solution $U \in C^0([0,T), H^k \times H^{k-2})$, but due to $\Delta^2 u_0 \in H^{k-4}$ we have $f_\varepsilon(U) \in C^0([0,T), H^{k-4})$, which is insufficient for an application of Lemma 3.2 (and Lemma 3.3) in a fixed point argument for v.

We thus approximate u_0 by $u_0^\delta \in C^\infty(\mathbb{R}^n,\mathbb{R}^L)$ for $\delta > 0$ such that $\mathrm{supp}(\nabla u_0^\delta) \subset \mathbb{R}^n$ is compact with

$$u_0^\delta \to u_0 \;\; a.e., \quad \nabla u_0^\delta \to \nabla u_0 \;\text{ in } H^{k-1}(\mathbb{R}^n) \;\text{ as }\; \delta \to 0^+. \tag{3.17}$$

Hence $f_{\varepsilon,\delta}(U) \in C^0(0,T;H^{k-3}(\mathbb{R}^n))$ for $0 < T < \infty$ and where $f_{\varepsilon,\delta}$ is defined as above through u_0^δ. For the initial data $u_0^\delta,\ u_1$ we now prove the existence of a fixed point for the operator $v \mapsto \mathcal{S}(v)$ defined through

$$\begin{pmatrix} \mathcal{S}(v) \\ \tilde{\mathcal{S}}(v) \end{pmatrix} = T_\varepsilon(t) \begin{pmatrix} 0 \\ u_1 \end{pmatrix} + \int_0^t T_\varepsilon(t-s) \begin{pmatrix} 0 \\ f_{\varepsilon,\delta}(v) \end{pmatrix} ds, \tag{3.18}$$

$$\text{where} \quad v \in C^0([0,T), H^k) \cap C^1([0,T), H^{k-2}).$$

Thus, we define for $R > 0,\ T \in (0,1)$

$$\mathcal{B}_R(T) := \big\{ v \in C^0([0,T), H^k) \cap C^1([0,T), H^{k-2}) \mid ,\ v(0) = 0,\ v_t(0) = u_1,$$
$$\|v\|_{\mathcal{B}} := \|v_t\|_{L^\infty H^{k-2}} + \|v\|_{L^\infty L^2} + \big\|\nabla(v + u_0^\delta)\big\|_{L^\infty H^{k-1}} \le R \big\},$$

and

$$\|v_1 - v_2\|_{\mathcal{B}(T)} = \|v_1 - v_2\|_{L^\infty H^k} + \|\partial_t v_1 - \partial_t v_2\|_{L^\infty H^{k-2}},\ v_1, v_2 \in \mathcal{B}_R(T).$$

Let $\varepsilon \in (0,1]$ be fixed, $T \in (0,1)$ and $R > 0$. The map

$$\mathcal{S} : \mathcal{B}_R(T) \to \mathcal{B}_R(T)$$

is contractive (Lipschitz) with respect to $\|\cdot\|_{\mathcal{B}(T)}$ if we choose $R = R_\delta$ and $T = T_\delta$ with

$$R_\delta^k = 3(\big\|\nabla u_0^\delta\big\|_{H^{k-1}} + \|u_1\|_{H^{k-2}})^k =: 3R_{0,\delta}^k \ \text{ and}$$

$$T_\delta = \frac{1}{2} \min \left\{ \left(\frac{\sqrt[k]{3} - 1}{\sqrt[k]{3}} \right)^2 \frac{\varepsilon}{C^2(1 + 3R_{0,\delta}^k)^2}, \frac{\varepsilon}{C^2(1 + 6R_{0,\delta}^k)^2} \right\}.$$

For the proof of this statement, it suffices to prove the following estimates for $v,\ \tilde{v} \in \mathcal{B}_R(T)$

$$\|S(v)\|_{\mathcal{B}} \le \frac{C}{\varepsilon^{\frac{1}{2}}} T^{\frac{1}{2}} (1 + \|v\|_{\mathcal{B}}^k) \|v\|_{\mathcal{B}} + \big\|\nabla u_0^\delta\big\|_{H^{k-1}} + \|u_1\|_{H^{k-2}}, \quad \text{and} \tag{3.19}$$

$$\|S(v) - S(\tilde{v})\|_{\mathcal{B}(T)} \le \frac{C}{\varepsilon^{\frac{1}{2}}} T^{\frac{1}{2}} (1 + \|v\|_{\mathcal{B}}^k + \|\tilde{v}\|_{\mathcal{B}}^k) \|v - \tilde{v}\|_{\mathcal{B}(T)}. \tag{3.20}$$

By the use of the estimate (3.13) for $r = k - 3$, we need to estimate the norms

$$\big\|\mathcal{N}_\varepsilon(v + u_0^\delta)\big\|_{H^{k-3}}^2, \quad \text{and} \quad \big\|\mathcal{N}_\varepsilon(v + u_0^\delta) - \mathcal{N}_\varepsilon(\tilde{v} + u_0^\delta)\big\|_{H^{k-3}}^2.$$

This is done by the use of Lemma A.1 and Corollary A.4 combined with a careful application of the Moser estimate in Lemma 2.2. In fact we give more details below in Section 4 in the proof of the a priori estimate and in Section 6 for the uniqueness since this requires more thought.

We note that for $T_\delta > 0$ and R_δ as above we obtain in the fixed point $v^\delta = \mathcal{S}(v^\delta)$

(3.21)

$$\big\|v_t^\delta\big\|_{L^\infty H^{k-2}}^2 + \big\|v^\delta\big\|_{L^\infty H^k}^2 + \frac{\varepsilon}{2} \int_0^{T_\delta} \big\|v_t^\delta(s)\big\|_{H^{k-1}}^2 \ ds + \frac{\varepsilon}{2} \int_0^{T_\delta} \big\|\nabla(v^\delta + u_0^\delta)\big\|_{H^k}^2 \ ds \lesssim R_\delta^2.$$

Hence $v^\delta \in L^2([0,T_\delta), H^{k+1}) \cap H^1([0,T_\delta), H^{k-1})$ and we define $R_0,\ R,\ \tilde{T} > 0$ as above in the definition of $R_{0,\delta},\ R_\delta,\ T_\delta$ through $u_0,\ R_0$ and R respectively. Thus,

$$R_{0,\delta} \to R_0,\ R_\delta \to R,\ T_\delta \to \tilde{T},\ \text{as } \delta \to 0^+.$$

For $\delta > 0$ small enough, e.g. such that $T_\delta > \frac{1}{2}\tilde{T} =: T$ and $|R_{0,\delta} - R_0| \le R_0$, we have that $v^\delta : \mathbb{R}^n \times [0,T) \to \mathbb{R}^L$ is well defined and $\big\|v^\delta\big\|_{\mathcal{B}(T)} \le CR$ for a constant $C > 0$. We now argue that wlog $v^\delta \to v,\ \nabla(v^\delta + u_0^\delta) \to \nabla(v + u_0),\ \partial_t v^\delta \to \partial_t v$ strongly in $C^0([0,T), H^k),\ L^2([0,T), H^k)$ and $C^0([0,T), H^{k-2}) \cap L^2([0,T), H^{k-1})$, respectively. Here we note that for $\delta,\ \delta' > 0$ sufficiently small $v^\delta - v^{\delta'},\ \partial_t v^\delta - \partial_t v^{\delta'}$ solve (3.10) with nonlinearity

$$\mathcal{N}_\varepsilon(v^\delta + u_0^\delta) - \mathcal{N}_\varepsilon(v^{\delta'} + u_0^{\delta'}) + \Delta^2(v^\delta - v^{\delta'}) \in C^0([0,T), H^{k-3}).$$

Thus, by Lemma 3.3, we obtain (similar to the proof of the Lipschitz estimate (3.20)) the bound

$$\left\|v^\delta - v^{\delta'}\right\|^2_{\mathcal{B}(T)} + \frac{\varepsilon}{2}\int_0^{T_\delta} \left\|v_t^\delta(s) - v_t^{\delta'}(s)\right\|^2_{H^{k-1}} \, ds + \frac{\varepsilon}{2}\int_0^{T_\delta} \left\|\nabla(v^\delta - v^{\delta'}) + \nabla(u_0^\delta - u_0^{\delta'})\right\|^2_{H^k} \, ds$$
$$\leq C\frac{T}{\varepsilon}(1+R^{2k})\left\|v^\delta - v^{\delta'}\right\|^2_{\mathcal{B}(T)} + \tilde{C}_{\varepsilon,R}\left\|\nabla u_0^\delta - \nabla u_0^{\delta'}\right\|^2_{H^{k-1}}.$$

Hence, if T is sufficiently small, then we deduce strong convergence and denote the δ-limit of v^δ by

$$v \in C^0([0,T),H^k) \cap C^1([0,T),H^{k-2}) \cap H^1((0,T),H^{k-1})$$

with $\nabla(v+u_0) \in L^2([0,T),H^k)$ .Thus, in particular, v, v_t solve (3.7) and $u = v + u_0$ solves (3.3). Further (3.6) holds for $u^\delta = v^\delta + u_0^\delta$ and we conclude the estimate for u since in particular $u_t^\delta \to u_t$ strongly in $C^0([0,T),H^{k-2})$ and $\mathcal{N}_\varepsilon(u^\delta) \to \mathcal{N}_\varepsilon(u)$ strongly in $C^0([0,T),H^{k-3})$ by the use of Corollary A.4 and Lemma 2.2 as above.

For the uniqueness of v, we note that if $\tilde{v}$ is a second solution, then $w = v - \tilde{v}$, $w_t = v_t - \tilde{v}_t$ solve (3.10) with the nonlinearity $\mathcal{N}_\varepsilon(v+u_0) - \mathcal{N}(\tilde{v}+u_0) \in C^0([0,T),H^{k-3})$. Hence, we use Lemma 3.3 (note that u_0 from the Lemma is different, namely $u_0 = 0$), in order to prove the estimate

$$\|v-\tilde{v}\|^2_{\mathcal{B}(T)} \leq C\frac{T}{\varepsilon}(1+R^{2k})\|v-\tilde{v}\|^2_{\mathcal{B}(T)}. \tag{3.22}$$

Hence, if T is sufficiently small, then $v = \tilde{v}$ and thus $u = v + u_0$ is unique. □

Next we show that the solution we just constructed actually takues values in the target manifold.

Proposition 3.4. Let $\varepsilon \in (0,1)$, $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}^L$ with $u_0(x) \in N$ and $u_1(x) \in T_{u_0(x)}N$ for a.e. $x \in \mathbb{R}^n$, and such that

$$(\nabla u_0, u_1) \in H^{k-1}(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n)$$

for some $k \in \mathbb{N}$ with $k > \lfloor \frac{n}{2} \rfloor + 2$. Then there exists a $T > 0$ such that the unique solution $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}^L$ of (3.1), which we constructed in Lemma 3.1, takes values in N, i.e., $u : \mathbb{R}^n \times [0,T) \to N$.

Proof. Let $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}^L$ be the solution constructed in Lemma 3.1. In the following we show that $u(x,t) \in N$ for $x \in \mathbb{R}^n$ and $t > 0$ small enough. Since

$$C^0([0,T),H^k) \hookrightarrow C^0(\mathbb{R}^n \times [0,T)),$$

and $u_0 \in N$ a.e. on $\mathbb{R}^n$ there exists $\tilde{T} \in (0,T]$ such that for $t \in [0,\tilde{T})$

$$\|\operatorname{dist}(u(t),N)\|_{L^\infty} \leq \sup_{x\in\mathbb{R}^n} |u(x,t) - u_0(x)| \lesssim \|u(t) - u_0\|_{H^k}$$

is sufficiently small for $\bar{u} = \pi(u)$ being well-defined. Next we let $w = \bar{u} - u$ and we note that $w(0) = \partial_t w(0) = 0$. Then we calculate

$$\begin{aligned}
\partial_t^2 \bar{u} =& d\pi_u \partial_t^2 u + d^2\pi_u(u_t,u_t),\\
\Delta \bar{u}_t =& d\pi_u \Delta u_t + d^2\pi_u(\Delta u, u_t) + 2d^2\pi_u(\nabla u_t, \nabla u) + d^3\pi_u(\nabla u, \nabla u, u_t),\\
\Delta^2 \bar{u} =& d\pi_u \Delta^2 u + d^2\pi_u(\Delta u, \Delta u) + 4d^2\pi_u(\nabla u, \nabla \Delta u) + 2d^2\pi_u(\nabla^2 u, \nabla^2 u)\\
&+ 2d^3\pi_u(\nabla u, \nabla u, \Delta u) + 4d^3\pi_u(\nabla u, \nabla u, \nabla^2 u)\\
&+ d^4\pi_u(\nabla u, \nabla u, \nabla u, \nabla u)
\end{aligned}$$

and hence we conclude that

$$\begin{aligned}
(\partial_t^2 + \Delta^2 - \varepsilon\Delta\partial_t)w =& d\pi_u\Big((\partial_t^2 + \Delta^2 - \varepsilon\Delta\partial_t)u\Big) + \mathcal{N}_\varepsilon(u) - \mathcal{N}_\varepsilon(u)\\
=& d\pi_u(\mathcal{N}_\varepsilon(u)) \in T_{\bar{u}}N.
\end{aligned}$$

Next we note that

$$w_t = \Big((\pi - I)(u)\Big)_t = (d\pi_{\bar u} - I)u_t \perp T_{\bar u}N$$

and thus by testing the above equation for w by w_t we obtain

$$\partial_t \frac{1}{2}\int_{\mathbb{R}^n} |w_t|^2\, dx + \partial_t \frac{1}{2}\int_{\mathbb{R}^n} |\Delta w|^2\, dx + \varepsilon \int_{\mathbb{R}^n} |\nabla w_t|^2\, dx = 0.$$

This implies that $w_t = \Delta w \equiv 0$ and therefore

$$\partial_t \int_{\mathbb{R}^n} |\nabla w|^2\, dx \leq 2\|\partial_t w\|_{L^2}\|\Delta w\|_{L^2} = 0$$

which in turn shows that $\nabla w \equiv 0$. Altogether this implies $w \equiv 0$ and therefore $u \in N$. $\square$

Remark 3.5. We remark that up to now we fixed $\varepsilon \in (0,1)$. Since the constants in the upper bound in estimates such as (3.21) are of order $O\left(\varepsilon^{-1}\right)$ we have to prove ε independent estimates in the next section.

4. A priori estimate

We now prove an a priori estimate for the solution $u : \mathbb{R}^n \times [0, T_\varepsilon) \to N$ of the equation

$$\partial_t^2 u + \Delta^2 u - \varepsilon \Delta \partial_t u \perp T_u N, \quad \text{on } \mathbb{R}^n \times [0,T), \tag{4.1}$$

given by Proposition 3.4 with $\varepsilon \in (0,1)$ and initial data $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}^L$, $u_0(x) \in N$, $u_1(x) \in T_{u_0(x)}N$, for a.e. $x \in \mathbb{R}^n$ and such that

$$(\nabla u_0, u_1) \in H^{k-1}(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n)$$

for some $k \in \mathbb{N}$, $k > \lfloor \frac{n}{2} \rfloor + 2$. We recall though $u = u^\varepsilon$ and $T = T_\varepsilon$ we write (u,T) instead of $(u^\varepsilon, T_\varepsilon)$ for the moment.

We recall that (3.6) holds for the solution u in Proposition 3.4, i.e. for $t \in [0,T]$.

$$\begin{aligned}
&\left\|\nabla^{k-2} u_t(t)\right\|_{L^2}^2 + \left\|\nabla^k u(t)\right\|_{L^2}^2 + \varepsilon \int_0^t \left\|\nabla^{k-1} u_t(s)\right\|_{L^2}^2\, ds \\
&\leq \int_0^t \int_{\mathbb{R}^n} \nabla^{k-2}\left[\mathcal{N}(u) - \varepsilon(I - P_u)(\Delta u_t)\right] \cdot \nabla^{k-2} u_t\, dx\, ds \\
&\quad + \left\|\nabla^{k-2} u_1\right\|_{L^2}^2 + \left\|\nabla^k u_0\right\|_{L^2}^2 .
\end{aligned} \tag{4.2}$$

In the following, we make use of the fact that $\mathcal{N}(u) \perp T_u N$ since $u(x,t) \in N$ for a.e. $(x,t) \in \mathbb{R}^n \times [0,T)$.

We thus first write (Note $(I - P_u)^2 = I - P_u$, ie. $\mathcal{N}(u) = (I - P_u)\mathcal{N}(u)$)

$$\begin{aligned}
\nabla^{k-2}(\mathcal{N}(u))\nabla^{k-2}u_t &= \sum_{\substack{m_1+m_2=k-2\\ m_1>0}} \nabla^{m_1}(I - P_u)\nabla^{m_2}(\mathcal{N}(u))\nabla^{k-2}u_t \\
&\quad + \nabla^{k-2}(\mathcal{N}(u))(I - P_u)\nabla^{k-2}u_t \\
&= \sum_{\substack{m_1+m_2=k-2\\ m_1>0}} \nabla^{m_1}(I - P_u)\nabla^{m_2}(\mathcal{N}(u))\nabla^{k-2}u_t \\
&\quad - \sum_{\substack{l_1+l_2=k-2\\ l_1>0}} \nabla^{k-2}(\mathcal{N}(u))\nabla^{l_1}[(I - P_u)]\nabla^{l_2}u_t \\
&=: I_1 + I_2,
\end{aligned}$$

where for the last equality, we used the Leibniz formula

$$0 = \nabla^{k-2}\left[(I - P_u)u_t\right] = \sum_{\substack{l_1+l_2=k-2\\ l_1>0}} \nabla^{l_1}[(I - P_u)]\nabla^{l_2}u_t + (I - P_u)\nabla^{k-2}u_t. \tag{4.3}$$

For I_1, we note that with $m_1 \in \mathbb{N}$ from Lemma 2.1

$$\nabla^{m_1}(I - P_u) = -\sum_{j=1}^{m_1} \sum_{\sum_i^j \tilde{k}_i = m_1 - j} d^j P_u(\nabla^{\tilde{k}_1+1} u \star \cdots \star \nabla^{\tilde{k}_j+1} u), \tag{4.4}$$

which implies the pointwise bound

$$|\nabla^{m_1}(I - P_u)| \lesssim \sum_{j=1}^{m_1} \sum_{\sum_i^j \tilde{k}_i = m_1 - j} |\nabla^{\tilde{k}_1+1} u| \cdots |\nabla^{\tilde{k}_j+1} u|. \tag{4.5}$$

Also, using Lemma A.1, $|\nabla^{m_2}(\mathcal{N}(u))|$ is pointwise bounded (up to a constant) by terms of the form

$$|\nabla^{\tilde{m}_1+1} u| \cdots |\nabla^{\tilde{m}_i} u| \big[|\nabla^{k_1} u_t||\nabla^{k_2} u_t| + |\nabla^{k_1+2} u||\nabla^{k_2+2} u| + |\nabla^{k_1+3} u||\nabla^{k_2+1} u| \big], \tag{4.6}$$

$$|\nabla^{\tilde{m}_1+1} u| \cdots |\nabla^{\tilde{m}_i} u| \big[|\nabla^{k_1+1} u||\nabla^{k_2+1} u||\nabla^{k_3+2} u| \big], \quad \text{and} \tag{4.7}$$

$$|\nabla^{\tilde{m}_1+1} u| \cdots |\nabla^{\tilde{m}_i} u| \big[|\nabla^{k_1+1} u||\nabla^{k_2+1} u||\nabla^{k_3+1} u||\nabla^{k_4+1} u| \big], \tag{4.8}$$

where $i = 1, \ldots, m_2,\ \tilde{m}_1 + \cdots + \tilde{m}_i + k_1 + k_2 = m_2 - i,\ \tilde{m}_1 + \cdots + \tilde{m}_i + k_1 + k_2 + k_3 = m_2 - i,\ \tilde{m}_1 + \cdots + \tilde{m}_i + k_1 + k_2 + k_3 + k_4 = m_2 - i$, respectively or (in the case $i = 0$)

$$|\nabla^{k_1} u_t||\nabla^{k_2} u_t| + |\nabla^{k_1+2} u||\nabla^{k_2+2} u| + |\nabla^{k_1+3} u||\nabla^{k_2+1} u|, \tag{4.9}$$

$$|\nabla^{k_1+1} u||\nabla^{k_2+1} u||\nabla^{k_3+2} u|, \quad \text{and} \tag{4.10}$$

$$|\nabla^{k_1+1} u||\nabla^{k_2+1} u||\nabla^{k_3+1} u||\nabla^{k_4+1} u|, \tag{4.11}$$

where $k_1 + k_2 = m_2,\ k_1 + k_2 + k_3 = m_2,\ k_1 + k_2 + k_3 + k_4 = m_2$, respectively. Note here that since $m_1 > 0$, we have $m_2 \leq k - 3$ and we use all bounds in the notation (4.6) - (4.8) above, where for the latter three cases we set $i = 0$.

We split

$$\begin{aligned}
&\int_{\mathbb{R}^n} \nabla^{k-2}(\mathcal{N}(u) - \varepsilon(I - P_u)(\Delta u_t)) \cdot \nabla^{k-2} u_t \, dx \\
&\qquad = \int_{\mathbb{R}^n} \nabla^{k-2}(\mathcal{N}(u)) \cdot \nabla^{k-2} u_t \, dx - \varepsilon \int_{\mathbb{R}^n} \nabla^{k-2}((I - P_u)(\Delta u_t)) \cdot \nabla^{k-2} u_t \, dx \\
&\qquad = \int_{\mathbb{R}^n} I_1 \, dx + \int_{\mathbb{R}^n} I_2 \, dx - \varepsilon \int_{\mathbb{R}^n} \nabla^{k-2}((I - P_u)(\Delta u_t)) \cdot \nabla^{k-2} u_t \, dx,
\end{aligned}$$

and start by estimating

$$\int_{\mathbb{R}^n} I_1 \, dx \leq \sum_{\substack{m_1 + m_2 = k-2 \\ m_1 > 0}} \|\nabla^{m_1}(I - P_u) \nabla^{m_2}(\mathcal{N}(u))\|_{L^2} \left\|\nabla^{k-2} u_t\right\|_{L^2}.$$

Hence, we continue with Lemma 2.2 in order to estimate the norm

$$\|\nabla^{m_1}(I - P_u) \nabla^{m_2}(\mathcal{N}(u))\|_{L^2},$$

in the following cases.

<u>*Case 1:*</u> $\nabla^{k_1} u_t \star \nabla^{k_2} u_t$

We use Lemma 2.2 with

$$f_1 = \nabla u,\ \ldots,\ f_j = \nabla u,\ f_{j+1} = \nabla u,\ \ldots,\ f_{j+i} = \nabla u,\ f_{j+i+1} = u_t,\ f_{j+i+2} = u_t,$$

and derivatives of order

$$\tilde{k}_1 + \cdots + \tilde{k}_j + \tilde{m}_1 + \cdots + \tilde{m}_i + k_1 + k_2 = m_1 + m_2 - i - j = k - 2 - (i + j).$$

Hence

(4.12)

$$
\begin{aligned}
&\left\| |\nabla^{\tilde{k}_1+1}u|\cdots|\nabla^{\tilde{k}_j+1}u||\nabla^{\tilde{m}_1+1}u|\cdots|\nabla^{\tilde{m}_i+1}u||\nabla^{k_1}u_t||\nabla^{k_2}u_t|\right\|_{L^2}\\
&\lesssim (1+\|u_t\|_{L^\infty}^2+\|\nabla u\|_{L^\infty}^{k-3}\|u_t\|_{L^\infty}^2+\|\nabla u\|_{L^\infty}^{k-2}\|u_t\|_{L^\infty})(\|\nabla u\|_{H^{k-2-i-j}}+\|u_t\|_{H^{k-2-i-j}})\\
&\lesssim (1+\|\nabla u\|_{L^\infty}^{k-1}+\|u_t\|_{L^\infty}^{k-1})(\|\nabla u\|_{H^{k-1}}+\|u_t\|_{H^{k-2}}),
\end{aligned}
$$

with Young's inequality in the latter estimate. For the other cases, we use Lemma 2.2 in a similar way.

<u>*Case 2:*</u> $\nabla^{k_1+2}u\star\nabla^{k_2+2}u$

We use Lemma 2.2 with

$$f_1=\nabla u,\ \ldots,\ f_j=\nabla u,\ f_{j+1}=\nabla u,\ \ldots,\ f_{j+i}=\nabla u,\ f_{j+i+1}=\nabla^2 u,\ f_{j+i+2}=\nabla^2 u,$$

and derivatives of order

$$\tilde{k}_1+\cdots+\tilde{k}_j+\tilde{m}_1+\cdots+\tilde{m}_i+k_1+k_2=m_1+m_2-i-j=k-2-(i+j).$$

Hence, we estimate

(4.13)

$$
\begin{aligned}
&\left\| |\nabla^{\tilde{k}_1+1}u|\cdots|\nabla^{\tilde{k}_j+1}u||\nabla^{\tilde{m}_1+1}u|\cdots|\nabla^{\tilde{m}_i+1}u||\nabla^{k_1+2}u||\nabla^{k_2+2}u|\right\|_{L^2}\\
&\lesssim (1+\left\|\nabla^2 u\right\|_{L^\infty}^2+\|\nabla u\|_{L^\infty}^{k-3}\left\|\nabla^2 u\right\|_{L^\infty}^2+\|\nabla u\|_{L^\infty}^{k-2}\left\|\nabla^2 u\right\|_{L^\infty})(\|\nabla u\|_{H^{k-2-i-j}}+\left\|\nabla^2 u\right\|_{H^{k-2-i-j}})\\
&\lesssim (1+\|\nabla u\|_{L^\infty}^{k-1}+\left\|\nabla^2 u\right\|_{L^\infty}^{k-1})(\|\nabla u\|_{H^{k-1}}+\left\|\nabla^2 u\right\|_{H^{k-2}}),
\end{aligned}
$$

<u>*Case 3:*</u> $\nabla^{k_1+3}u\star\nabla^{k_2+1}u$

Here, the cancellation from (4.3) is exploited. We use Lemma 2.2 with

$$f_1=\nabla u,\ \ldots,\ f_j=\nabla u,\ f_{j+1}=\nabla u,\ \ldots,\ f_{j+i}=\nabla u,\ f_{j+i+1}=\nabla^2 u,\ f_{j+i+2}=\nabla u,$$

and derivatives of order

$$\tilde{k}_1+\cdots+\tilde{k}_j+\tilde{m}_1+\cdots+\tilde{m}_i+k_1+1+k_2=m_1+m_2+1-i-j=k-1-(i+j).$$

Hence, noting $j\geq 1$ since $m_1>0$,

(4.14)

$$
\begin{aligned}
&\left\| |\nabla^{\tilde{k}_1+1}u|\cdots|\nabla^{\tilde{k}_j+1}u||\nabla^{\tilde{m}_1+1}u|\cdots|\nabla^{\tilde{m}_i+1}u||\nabla^{k_1+3}u||\nabla^{k_2+1}u|\right\|_{L^2}\\
&\lesssim (1+\left\|\nabla^2 u\right\|_{L^\infty}+\|\nabla u\|_{L^\infty}^{k-2}\left\|\nabla^2 u\right\|_{L^\infty}+\|\nabla u\|_{L^\infty}^{k-1})(\|\nabla u\|_{H^{k-1-i-j}}+\left\|\nabla^2 u\right\|_{H^{k-1-i-j}})\\
&\lesssim (1+\|\nabla u\|_{L^\infty}^{k-1}+\left\|\nabla^2 u\right\|_{L^\infty}^{k-1})(\|\nabla u\|_{H^{k-1}}+\left\|\nabla^2 u\right\|_{H^{k-2}}),
\end{aligned}
$$

<u>*Case 4:*</u> $\nabla^{k_1+1}u\star\nabla^{k_2+1}u\star\nabla^{k_3+2}u$

We use Lemma 2.2 with

$$f_1=\nabla u,\ \ldots,\ f_j=\nabla u,\ f_{j+1}=\nabla u,\ \ldots,\ f_{j+i}=\nabla u,\ f_{j+i+1}=\nabla u,\ f_{j+i+2}=\nabla u,\ f_{j+i+3}=\nabla^2 u$$

and derivatives of order

$$\tilde{k}_1+\cdots+\tilde{k}_j+\tilde{m}_1+\cdots+\tilde{m}_i+k_1+k_2+k_3=m_1+m_2-i-j=k-2-(i+j).$$

Hence, we have

(4.15)

$$
\begin{aligned}
&\left\| |\nabla^{\tilde{k}_1+1}u|\cdots|\nabla^{\tilde{k}_j+1}u||\nabla^{\tilde{m}_1+1}u|\cdots|\nabla^{\tilde{m}_i+1}u||\nabla^{k_1+1}u||\nabla^{k_2+1}u||\nabla^{k_3+2}u|\right\|_{L^2}\\
&\lesssim (1+\left\|\nabla^2 u\right\|_{L^\infty}+\|\nabla u\|_{L^\infty}^{k}+\|\nabla u\|_{L^\infty}^{k-1}\left\|\nabla^2 u\right\|_{L^\infty})(\|\nabla u\|_{H^{k-2-i-j}}+\left\|\nabla^2 u\right\|_{H^{k-2-i-j}})\\
&\lesssim (1+\|\nabla u\|_{L^\infty}^{k}+\left\|\nabla^2 u\right\|_{L^\infty}^{k})(\|\nabla u\|_{H^{k-1}}+\left\|\nabla^2 u\right\|_{H^{k-2}}),
\end{aligned}
$$

<u>*Case 5:*</u> $\nabla^{k_1+1}u\star\nabla^{k_2+1}u\star\nabla^{k_3+1}u\star\nabla^{k_4+1}u$

We use Lemma 2.2 with

$$f_1 = \nabla u,\ \dots,\ f_j = \nabla u,\ f_{j+1} = \nabla u,\ \dots,\ f_{j+i} = \nabla u,\ f_{j+i+1} = \nabla u,$$
$$f_{j+i+2} = \nabla u,\ f_{j+i+3} = \nabla u,\ f_{j+i+4} = \nabla u$$

and derivatives of order

$$\tilde{k}_1 + \cdots + \tilde{k}_j + \tilde{m}_1 + \cdots + \tilde{m}_i + k_1 + k_2 + k_3 + k_4 = m_1 + m_2 - i - j = k - 2 - (i + j).$$

Hence, we have

$$\begin{aligned}
&\left\| |\nabla^{\tilde{k}_1+1}u| \cdots |\nabla^{\tilde{k}_j+1}u||\nabla^{\tilde{m}_1+1}u| \cdots |\nabla^{\tilde{m}_i+1}u||\nabla^{k_1+1}u||\nabla^{k_2+1}u||\nabla^{k_3+1}u||\nabla^{k_4+1}u| \right\|_{L^2} \\
&\lesssim (1 + \|\nabla u\|_{L^\infty}^{k+1}) \|\nabla u\|_{H^{k-2-i-j}} \\
&\lesssim (1 + \|\nabla u\|_{L^\infty}^{k+1}) \|\nabla u\|_{H^{k-1}}\ .
\end{aligned} \tag{4.16}$$

Now, for estimating I_2, we integrate by parts in order to conclude

$$\begin{aligned}
\int_{\mathbb{R}^n} I_2\ dx &= \sum_{\substack{l_1+l_2=k-2 \\ l_1>0}} \int_{\mathbb{R}^n} \nabla^{k-3}(\mathcal{N}(u)) \cdot [\nabla^{l_1+1}(I - P_u)\nabla^{l_2} u_t + \nabla l_1 (I - P_u)\nabla^{l_2+1} u_t]\ dx \\
&= \sum_{\substack{l_1+l_2=k-2 \\ l_1>0}} \int_{\mathbb{R}^n} \nabla^{k-3}(\mathcal{N}(u)) \cdot [\nabla^{l_1+1}(I - P_u)\nabla^{l_2} u_t]\ dx \\
&\quad + \sum_{\substack{l_1+l_2=k-2 \\ l_1>0}} \int_{\mathbb{R}^n} \nabla^{k-3}(\mathcal{N}(u)) \cdot [\nabla^{l_1}(I - P_u)\nabla^{l_2+1} u_t]\ dx \\
&=: I_2^1 + I_2^2.
\end{aligned}$$

Both terms are estimated by

$$\left\|I_2^1\right\|_{L^1} \leq \sum_{\substack{l_1+l_2=k-2 \\ l_1>0}} \left\|\nabla^{k-3}(\mathcal{N}(u))\right\|_{L^2} \left\|\nabla^{l_1+1}(I - P_u)\nabla^{l_2} u_t\right\|_{L^2} \tag{4.17}$$

$$\left\|I_2^2\right\|_{L^1} \leq \sum_{\substack{l_1+l_2=k-2 \\ l_1>0}} \left\|\nabla^{k-3}(\mathcal{N}(u))\right\|_{L^2} \left\|\nabla^{l_1}(I - P_u)\nabla^{l_2+1} u_t\right\|_{L^2} \tag{4.18}$$

We estimate $\left\|\nabla^{k-3}(\mathcal{N}(u))\right\|_{L^2}$ by terms of the form (4.6) - (4.8) in the L^2 norm. Then, estimating these norms using Lemma 2.2 is very similar to the case by case analysis above and we note the bound

$$\left\|\nabla^{k-3}(\mathcal{N}(u))\right\|_{L^2} \lesssim (1 + \|\nabla u\|_{L^\infty}^{k} + \|u_t\|_{L^\infty}^{k-2})(\|\nabla u\|_{H^{k-1}} + \|u_t\|_{H^{k-2}}). \tag{4.19}$$

Thus, it remains to estimate (again the cancellation (4.3) is important here)

$$\begin{aligned}
\left\|\nabla^{l_1+1}(I - P_u)\nabla^{l_2} u_t\right\|_{L^2} &= \left\| |\nabla^{\tilde{m}_1+1}u| \cdots |\nabla^{\tilde{m}_i+1}u||\nabla^{l_2}u_t| \right\|_{L^2} \\
&\lesssim (1 + \|\nabla u\|_{L^\infty}^{k-1} + \|u_t\|_{L^\infty}^{k-1})(\|\nabla u\|_{H^{k-1}} + \|u_t\|_{H^{k-2}})
\end{aligned} \tag{4.20}$$

by Lemma 2.2 with

$$\tilde{m}_1 + \cdots + \tilde{m}_i + l_2 = k - 1 - i \leq k - 2, \quad (l_1 > 0,\ \text{hence } i > 0).$$

Similarly, we have

$$\begin{aligned}
\left\|\nabla^{l_1}(I - P_u)\nabla^{l_2+1} u_t\right\|_{L^2} &= \left\| |\nabla^{\tilde{m}_1+1}u| \cdots |\nabla^{\tilde{m}_i+1}u||\nabla^{l_2+1}u_t| \right\|_{L^2} \\
&\lesssim (1 + \|\nabla u\|_{L^\infty}^{k-2} + \|u_t\|_{L^\infty}^{k-2})(\|\nabla u\|_{H^{k-1}} + \|u_t\|_{H^{k-2}})
\end{aligned} \tag{4.21}$$

by Lemma 2.2 with

$$\tilde{m}_1 + \cdots + \tilde{m}_i + l_2 + 1 = k - 1 - i \leq k - 2, \quad (l_1 > 0).$$

This implies

$$\|I_2\|_{L^1} \lesssim (1 + \|\nabla u\|_{L^\infty}^{2(k-1)} + \|u_t\|_{L^\infty}^{2(k-1)})(\|\nabla u\|_{H^{k-1}}^2 + \|u_t\|_{H^{k-2}}^2). \tag{4.22}$$

Finally, for the regularization term, we observe

$$\begin{aligned}-\varepsilon \int_{\mathbb{R}^n} \nabla^{k-2}[(I-P_u)(\Delta u_t)]\nabla^{k-2}u_t \; dx = \; & \varepsilon \int_{\mathbb{R}^n} \nabla^{k-3}[(I-P_u)(\Delta u_t)]\nabla^{k-1}u_t \; dx \\ \leq & \; C \left\|\nabla^{k-3}[(I-P_u)(\Delta u_t)]\right\|_{L^2}^2 + \frac{\varepsilon}{2}\left\|\nabla^{k-1}u_t\right\|_{L^2}^2 .\end{aligned}$$

Thus, in order to bound the norm $\left\|\nabla^{k-3}[(I-P_u)(\Delta u_t)]\right\|_{L^2}^2$, by (3.3) it suffices to estimate

$$\left\||\nabla^{\tilde m_1+1}u|\cdots|\nabla^{\tilde m_i+1}u|\left[|\nabla^{k_1+1}u_t||\nabla^{k_2+1}u| + |\nabla^{k_1}u_t||\nabla^{k_2+2}u|\right]\right\|_{L^2}^2, \tag{4.23}$$

$$\left\||\nabla^{\tilde m_1+1}u|\cdots|\nabla^{\tilde m_i+1}u||\nabla^{k_1}u_t||\nabla^{k_2+1}u||\nabla^{k_3+1}u|\right\|_{L^2}^2, \tag{4.24}$$

where $\tilde m_1 + \cdots + \tilde m_i + k_1 + k_2 = k-3-i$, $\tilde m_1 + \cdots + \tilde m_i + k_1 + k_2 + k_3 = k-3-i$, respectively.

By the Lemma 2.2, we have the estimates

$$\begin{aligned}&\left\||\nabla^{\tilde m_1+1}u|\cdots|\nabla^{\tilde m_i+1}u|\left[|\nabla^{k_1+1}u_t||\nabla^{k_2+1}u| + |\nabla^{k_1}u_t||\nabla^{k_2+2}u|\right]\right\|_{L^2}^2, \\ &\lesssim (1 + \|\nabla u\|_{L^\infty}^{2(k-2)} + \left\|\nabla^2 u\right\|_{L^\infty}^{2(k-2)} + \|u_t\|_{L^\infty}^{2(k-2)})(\|u_t\|_{H^{k-2}}^2 + \|\nabla u\|_{H^{k-2}}^2), \quad \text{and}\end{aligned} \tag{4.25}$$

$$\begin{aligned}&\left\||\nabla^{\tilde m_1+1}u|\cdots|\nabla^{\tilde m_i+1}u||\nabla^{k_1}u_t||\nabla^{k_2+1}u||\nabla^{k_3+1}u|\right\|_{L^2}^2, \\ &\lesssim (1 + \|\nabla u\|_{L^\infty}^{2(k-1)} + \|u_t\|_{L^\infty}^{2(k-1)})(\|u_t\|_{H^{k-2}}^2 + \|\nabla u\|_{H^{k-2}}^2).\end{aligned} \tag{4.26}$$

Thus, putting together (4.12) - (4.16), (4.19)- (4.21), (4.25) and (4.26), we estimate

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} \nabla^{k-2}(\mathcal{N}(u) - \varepsilon(I-P_u)(\Delta u_t)) \cdot \nabla^{k-2}u_t \; dx\right| \\ &\leq (1 + \|\nabla u\|_{L^\infty}^{2k} + \left\|\nabla^2 u\right\|_{L^\infty}^{2k} + \|u_t\|_{L^\infty}^{2k})(\|\nabla u\|_{H^{k-1}}^2 + \|u_t\|_{H^{k-2}}^2) + \frac{\varepsilon}{2}\left\|\nabla^{k-1}u_t\right\|_{L^2}^2 .\end{aligned}$$

Hence, subtracting

$$\frac{\varepsilon}{2}\int_0^t \left\|\nabla^{k-1}u_t(s)\right\|_{L^2}^2 \; ds,$$

on both sides of (4.2), we have for $t \in [0,T)$

$$\begin{aligned}&\left\|\nabla^{k-2}u_t(t)\right\|_{L^2}^2 + \left\|\nabla^k u(t)\right\|_{L^2}^2 + \frac{\varepsilon}{2}\int_0^t \left\|\nabla^{k-1}u_t(s)\right\|_{L^2}^2 \; ds \\ &\lesssim \int_0^t \left[(1 + \|\nabla u\|_{L^\infty}^{2k} + \left\|\nabla^2 u\right\|_{L^\infty}^{2k} + \|u_t\|_{L^\infty}^{2k})(\|\nabla u\|_{H^{k-1}}^2 + \|u_t\|_{H^{k-2}}^2)\right] \; ds \\ &\quad + \left\|\nabla^{k-2}u_1\right\|_{L^2}^2 + \left\|\nabla^k u_0\right\|_{L^2}^2 .\end{aligned} \tag{4.27}$$

Since, testing (4.1) by $u_t \in T_uN$ for $t \in [0,T)$ also gives

$$\|u_t(t)\|_{L^2}^2 + \|\Delta u(t)\|_{L^2}^2 + \varepsilon\int_0^t \|\nabla u_t(s)\|_{L^2}^2 \; ds = \|u_1\|_{L^2}^2 + \|\Delta u_0\|_{L^2}^2 ,$$

we thus use

$$\frac{d}{dt}\int_{\mathbb{R}^n} |\nabla u|^2 \; dx \leq \int_{\mathbb{R}^n} |u_t|^2 \; dx + \int_{\mathbb{R}^n} |\Delta u|^2 \; dx,$$

and interpolation of lower order derivatives in the L^2 norm on the left-hand side, more precisely

$$\left\|\nabla^l u_t\right\|_{L^2}^2 \lesssim \left\|\nabla^{k-1} u_t\right\|_{L^2}^{\frac{2(l-1)}{k-2}} \|\nabla u_t\|_{L^2}^{\frac{2(k-1-l)}{k-2}}, \ l = 2, \ldots, k-2, \tag{4.28}$$

$$\left\|\nabla^l u_t\right\|_{L^2}^2 \lesssim \left\|\nabla^{k-2} u_t\right\|_{L^2}^{\frac{2l}{k-2}} \|u_t\|_{L^2}^{\frac{2(k-2-l)}{k-2}}, \ l = 1, \ldots, k-3, \quad \text{and} \tag{4.29}$$

$$\left\|\nabla^l u\right\|_{L^2}^2 \lesssim \left\|\nabla^k u\right\|_{L^2}^{\frac{2(l-2)}{k-2}} \|\Delta u\|_{L^2}^{\frac{2(k-l)}{k-2}}, \ l = 3, \ldots, k-1, \tag{4.30}$$

in order to conclude

$$\begin{aligned} &\|u_t(t)\|_{H^{k-2}}^2 + \|\nabla u(t)\|_{H^{k-1}}^2 + \frac{\varepsilon}{2}\int_0^t \|\nabla u_t(s)\|_{H^{k-2}}^2 \ ds \\ &\lesssim \int_0^t \Big[(1 + \|\nabla u\|_{W^{1,\infty}}^{2k} + \|u_t\|_{L^\infty}^{2k})(\|\nabla u\|_{H^{k-1}}^2 + \|u_t\|_{H^{k-2}}^2)\Big] \ ds \\ &\quad + \|u_1\|_{H^{k-2}}^2 + \|\nabla u_0\|_{H^{k-1}}^2, \quad t \in [0, T). \end{aligned} \tag{4.31}$$

We remark that this implies for solutions of (3.1) by the Gronwall bound

$$\begin{aligned} &\sup_{t\in[0,T]} \Big(\|u_t(t)\|_{H^{k-2}}^2 + \|\nabla u(t)\|_{H^{k-1}}^2\Big) \\ &\leq C\Big(\|u_1\|_{H^{k-2}}^2 + \|\nabla u_0\|_{H^{k-1}}^2\Big) \exp\left(\int_0^T (1 + \|\nabla u\|_{W^{1,\infty}}^{2k} + \|u_t\|_{L^\infty}^{2k}) \ ds\right). \end{aligned} \tag{4.32}$$

We note that since the solutions to (3.1) are (locally) unique, the argument in the previous section of the derivation of (4.31) holds for $t_0 \in [0, T), \ t \in [t_0, T)$ with

$$\begin{aligned} &\|u_t(t)\|_{H^{k-2}}^2 + \|\nabla u(t)\|_{H^{k-1}}^2 + \frac{\varepsilon}{2}\int_{t_0}^t \|\nabla u_t(s)\|_{H^{k-2}}^2 \ ds \\ &\lesssim \int_{t_0}^t \Big[(1 + \|\nabla u\|_{W^{1,\infty}}^{2k} + \|u_t\|_{L^\infty}^{2k})(\|\nabla u\|_{H^{k-1}}^2 + \|u_t\|_{H^{k-2}}^2)\Big] \ ds \\ &\quad + \|u_t(t_0)\|_{H^{k-2}}^2 + \|\nabla u(t_0)\|_{H^{k-1}}^2. \end{aligned} \tag{4.33}$$

Thus, setting

$$\alpha(t) := \|u_t(t)\|_{H^{k-2}}^2 + \|\nabla u(t)\|_{H^{k-1}}^2, \quad t \in [0, T),$$

we obtain (for some constant $C > 0$)

$$\frac{d}{dt}\alpha(t) \leq C(1 + \alpha(t)^k)\,\alpha(t), \quad t \in [0, T). \tag{4.34}$$

We now proceed similar as in [KLP+10], where regularization by the (intrinsic) biharmonic energy has been applied in order to obtain the existence of local Schrödinger maps.

Lemma 4.1. *Let $\varepsilon \in (0, 1)$, $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}^L$, $u_0(x) \in N$, $u_1(x) \in T_{u_0(x)}N$, for a.e. $x \in \mathbb{R}^n$ and such that*

$$(\nabla u_0, u_1) \in H^{k-1}(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n)$$

for some $k \in \mathbb{N}$, $k > \lfloor \frac{n}{2} \rfloor + 2$. Then then there exists $T = T(\|\nabla u_0\|_{H^{k-1}}, \|u_1\|_{H^k}) > 0$ such that the solutions $(u^\varepsilon, T_\varepsilon)$ from Proposition 3.4 are solutions $u^\varepsilon : \mathbb{R}^n \times [0, T) \to N$.

Proof. We infer

$$\frac{d}{dt}\log\left(\frac{\alpha}{(1 + \alpha^k)^{\frac{1}{k}}}\right) = \frac{\alpha'}{(1 + \alpha^k)\,\alpha} \leq C, \tag{4.35}$$

which (by integration from 0 to t) gives for $\alpha_0 = \alpha(0)$

$$\frac{\alpha^k}{(1 + \alpha^k)} \leq e^{Ctk}\frac{\alpha_0^k}{(1 + \alpha_0^k)} \leq (1 + 4Ctk)\frac{\alpha_0^k}{(1 + \alpha_0^k)}, \quad 0 \leq t < \frac{1}{8Ck}.$$

Therefore

$$\alpha^k \le (1+4Ctk)\,\alpha_0^k + 4Ctk\,\alpha_0^k\,\alpha^k \quad \text{for } 0 \le t < \frac{1}{8Ck},$$

and thus

$$\alpha^k \le 2(1+4Ctk)\,\alpha_0^k \le 3\,\alpha_0^k, \quad \text{for } 0 \le t < \frac{1}{8Ck}\min\left\{1, \frac{1}{\alpha_0^k}\right\}.$$

Hence, setting $T_0 := \frac{1}{8Ck}\min\left\{1, \frac{1}{\alpha_0^k}\right\}$, we obtain

(4.36)

$$\|u_t(t)\|_{H^{k-2}}^2 + \|\nabla u(t)\|_{H^{k-1}}^2 \le \sqrt[k]{3}\,\|u_1\|_{H^{k-2}}^2 + \sqrt[k]{3}\,\|\nabla u_0\|_{H^{k-1}}^2\,, \quad t \in [0, \min\{T, T_0\}).$$

We recall that $u = u^\varepsilon$ (where $\varepsilon \in (0,1)$ is fixed) and we denote by T_ε the maximal existence time of u^ε.

We now assume by contradiction $T_\varepsilon < T_0$, where T_0 is taken from the previous section. Thus, applying the contraction argument in Section 3 for $t_0 \in [0, T_\varepsilon)$ in the space $\mathcal{B}_r(T)$ defined over $u(t_0)$ and with

$$r^k = 3r(t_0)^k = 3\Big(\,\|\nabla u(t_0)\|_{H^{k-1}} + 3\,\|u_t(t_0)\|_{H^{k-2}}\Big)^k,$$

we observe that there exists a constant $c > 0$ such that the solution will be uniquely extended to $[0, t_0 + T)$ as long as

$$T < c^{-1}\min\left\{\left(\frac{\sqrt[k]{3}-1}{\sqrt[k]{3}}\right)^2 \frac{\varepsilon}{C^2(1+3r(t_0)^k)^2}, \frac{\varepsilon}{C^2(1+6r(t_0)^k)^2}\right\}. \tag{4.37}$$

However, by (4.36) (note here $t_0 \in [0, T_0)$ by assumption), we succeed to solve (3.1) by the proof of Lemma 3.1 and Proposition 3.4 starting from $u(t_0),\ u_t(t_0)$ with the existence time

$$\begin{aligned} T &= \frac{1}{2c}\min\left\{\left(\frac{\sqrt[k]{3}-1}{\sqrt[k]{3}}\right)^2 \frac{\varepsilon}{C^2(1+9r_0^k)^2}, \frac{\varepsilon}{C^2(1+18r_0^k)^2}\right\} \\ &\le \frac{1}{2c}\min\left\{\left(\frac{\sqrt[k]{3}-1}{\sqrt[k]{3}}\right)^2 \frac{\varepsilon}{C^2(1+3r(t_0)^k)^2}, \frac{\varepsilon}{C^2(1+6r(t_0)^k)^2}\right\}, \end{aligned} \tag{4.38}$$

Since $T > 0$ does only depend on $u_0,\ u_1$ and hence not on the choice of $t_0 \in [0, T_\varepsilon)$, we infer a contradiction for $T_\varepsilon - t_0 < T$. Thus we set $T(u_0, u_1) = T_0$. □

5. Proof of the main theorem

We now combine the existence result from Lemma 3.1 and Proposition 3.4 with Lemma 4.1. Thus there exists a family of solutions $u^\varepsilon : \mathbb{R}^n \times [0,T) \to N$ of (3.1) for $\varepsilon \in (0,1)$, where $T = T(u_0, u_1)$ only depends on $u_0,\ u_1$. From (4.36) and the fact that

$$\|u^\varepsilon - u_0\|_{L^\infty L^2} \le T\,\|u_t^\varepsilon\|_{L^\infty L^2}\,,$$

we extract a limit $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}^L$ as $\varepsilon \to 0^+$ of the solutions $u^\varepsilon_{|[0,T)}$ in the sense

$$\nabla^{l_1} u^\varepsilon \overset{*}{\rightharpoonup} \nabla^{l_1} u, \quad u^\varepsilon - u_0 \overset{*}{\rightharpoonup} u - u_0, \quad \text{and } \nabla^{l_2-2} u_t^\varepsilon \overset{*}{\rightharpoonup} \nabla^{l_2-2} u_t \ \text{ in } \ L^\infty([0,T), L^2),$$

where $1 \le l_1 \le k$ and $0 \le l_2 \le k$. Thus we have

$$u - u_0 \in L^\infty([0,T), H^k) \cap W^{1,\infty}([0,T), H^{k-2}).$$

Further, by differentiating (3.1) up to order $k-4$ and estimating the nonlinearity similar as in section 4, we also obtain from (4.36) that $\partial_t^2 u^\varepsilon \in C^0([0,T), H^{k-4})$ is uniformely bounded as $\varepsilon \to 0^+$. By compactness and Sobolev's embedding, we further assume for $u^\varepsilon = v^\varepsilon + u_0$,

$$\nabla^3 u^\varepsilon \to \nabla^3 u, \quad \text{in} \ \ C^0([0,T), L^2_{loc}(\mathbb{R}^n))$$
$$\partial_t u^\varepsilon \to \partial_t u, \quad u^\varepsilon \to u, \ \nabla u^\varepsilon \to \nabla u, \ \nabla^2 u^\varepsilon \to \nabla^2 u, \ \ \text{locally uniformely on} \ \mathbb{R}^n \times [0,T_0).$$

More precisely for $\alpha \in (0,1)$ we have uniform bounds (in ε) in

$$v^\varepsilon \in C^\alpha H^{k-2\alpha}, \ \nabla v^\varepsilon \in C^\alpha H^{k-1-2\alpha}, \ \nabla^2 v^\varepsilon \in C^\alpha H^{k-2-2\alpha}, \ \partial_t v^\varepsilon \in C^\alpha H^{k-2-2\alpha}, \tag{5.1}$$

where the last fact follows from [Lun95, Prop. 1.1.4]. Thus there holds $u \in N$ on $\mathbb{R}^n \times [0,T)$ and since (4.31) combined with (4.36) gives

$$\begin{aligned}\int_0^T & \left\| \sqrt{\varepsilon} \nabla u_t^\varepsilon(s) \right\|^2_{H^{k-2}} \ ds \\ & \lesssim \big(T(1 + \|u_1\|^{2k}_{H^{k-2}} + \|\nabla u_0\|^{2k}_{H^{k-1}}) + 1\big)(\|u_1\|^2_{H^{k-2}} + \|\nabla u_0\|^2_{H^{k-1}}),\end{aligned} \tag{5.2}$$

we have $\varepsilon \Delta \partial_t u^\varepsilon \to 0$ in $L^2_{t,x}$ since $k \geq 3$. Also the coefficients in (1.3) converge (locally uniformly) and from the limits above, we see (considering the definition of $\mathcal{N}_\varepsilon$)

$$\mathcal{N}_\varepsilon(u^\varepsilon) \to \mathcal{N}(u) \ \ \text{in} \ L^2_{loc}(\mathbb{R}^n \times [0,T)).$$

Here we note in particular that $(I - P_{u^\varepsilon})(\Delta u_t^\varepsilon)$ converges in $L^2_{loc}(\mathbb{R}^n \times [0,T))$ as $\varepsilon \to 0^+$.

The blow-up criterion (1.4) follows from an energy estimate similar to (4.32) for biharmonic wave maps.

The uniqueness statement, which is left to conclude the proof of Theorem 1.1, is considered in the next section.

6. Uniqueness

Lemma 6.1. *Let $u, v : \mathbb{R}^n \times [0,T) \to N$ be two solutions of (1.2) with initial data $u_0 : \mathbb{R}^n \to N$, $u_1 : \mathbb{R}^n \to \mathbb{R}^L$ and $u_1 \in T_{u_0}N$ on $\mathbb{R}^n$ such that for some $k \in \mathbb{N}$ with $k > \lfloor \frac{n}{2} \rfloor + 2$ we have*

$$(\nabla u_0, u_1) \in H^{k-1}(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n).$$

Also let

$$u - u_0, \ v - u_0 \in L^\infty([0,T), H^k(\mathbb{R}^n)) \cap W^{1,\infty}([0,T), H^{k-2}(\mathbb{R}^n)).$$

Then $u_{|[0,T)} = v_{|[0,T)}$.

Proof of Lemma 6.1. We obtain the uniqueness from a Gronwall argument by estimating (with $w = u - v$)

$$\frac{d}{dt} \int_{\mathbb{R}^n} |\nabla^l w_t|^2 + |\nabla^{l+2} w|^2 \ dx = \int_{\mathbb{R}^n} \nabla^l(\mathcal{N}(u) - \mathcal{N}(v)) \cdot \nabla^l w_t \ dx, \tag{6.1}$$

for $l \in \{0, \dots, k-3\}$ and proving

$$\frac{d}{dt}\mathcal{E}^2(t) \leq C(1 + \|\nabla u\|^{2k}_{H^{k-1}} + \|u_t\|^{2k}_{H^{k-2}} + \|\nabla v\|^{2k}_{H^{k-1}} + \|v_t\|^{2k}_{H^{k-2}})\mathcal{E}^2(t), \quad t \in [0,T), \tag{6.2}$$

with

$$\mathcal{E}(t) = \|w(t)\|_{H^{k-1}} + \|w_t(t)\|_{H^{k-3}}, \ \ t \in [0,T).$$

We first prove an estimate for (6.1) in the case $l = k-3$, the case $l < k-3$ will be similar and in fact easier. We note that since u, v map to N, we have $\mathcal{N}(u) = (I - P_u)(\mathcal{N}(u))$. Thus, we write

$$\begin{aligned}\mathcal{N}(u) - \mathcal{N}(v) &= (I - P_u)\mathcal{N}(u) - (I - P_v)\mathcal{N}(v) \\ &= (P_v - P_u)\mathcal{N}(u) + (I - P_v)(\mathcal{N}(u) - \mathcal{N}(v)).\end{aligned}$$

Thus

$$\begin{aligned}\nabla^{k-3}(\mathcal{N}(u)-\mathcal{N}(v))\cdot\nabla^{k-3}w_t &= \nabla^{k-3}[(P_v-P_u)\mathcal{N}(u)]\cdot\nabla^{k-3}w_t\\ &\quad+\nabla^{k-3}[(I-P_v)(\mathcal{N}(u)-\mathcal{N}(v))]\cdot\nabla^{k-3}w_t.\end{aligned}$$

This is needed, in order to avoid the case where all derivatives fall on $\nabla^3 w$. Hence we write

$$\begin{aligned}\nabla^{k-3}[(P_v-P_u)\mathcal{N}(u)]\cdot\nabla^{k-3}w_t &= (P_v-P_u)\nabla^{k-3}[\mathcal{N}(u)]\cdot\nabla^{k-3}w_t\\ &\quad+\sum_{\substack{l_1+l_2=k-3\\ l_1>0}}\nabla^{l_1}[(P_v-P_u)]\nabla^{l_2}[\mathcal{N}(u)]\cdot\nabla^{k-3}w_t =: I_1+I_2.\end{aligned}$$

We note here

$$\int_{\mathbb{R}^n} I_1\ dx \lesssim \|w\|_{L^\infty}\left\|\nabla^{k-3}\mathcal{N}(u)\right\|_{L^2}\left\|\nabla^{k-3}w_t\right\|_{L^2},$$

where $\left\|\nabla^{k-3}\mathcal{N}(u)\right\|_{L^2}$ is estimated by the use of Lemma 2.2 as above in the a priori estimate. Further, Lemma 2.2 combined with Lemma A.2 also implies that $\int_{\mathbb{R}^n} I_2\ dx$ is bounded by terms of the form

$$\|w\|_{L^\infty}\left\||\nabla^{m_1+1}u|\cdots|\nabla^{m_j+1}u||\nabla^{l_2}\mathcal{N}(u)|\right\|_{L^2}\left\|\nabla^{k-3}w_t\right\|_{L^2}+ \tag{6.3}$$

$$\left\|\nabla^{k-3}w_t\right\|_{L^2}\left\||\nabla^{m_1+1}w||\nabla^{m_2+1}h_1|\cdots|\nabla^{m_j+1}h_{j-1}||\nabla^{l_2}\mathcal{N}(u)\right\|_{L^2}, \tag{6.4}$$

where $m_1,\dots,m_j,\ h_1,\dots,h_{j-1}$ are as in Lemma A.2. For (6.3) we then estimate as above in the a priori estimate and note for (6.4), it suffices to estimate terms of the form

$$|\nabla^{m_1+1}w||\nabla^{m_2+1}h_1|\cdots|\nabla^{m_j+1}h_{j-1}||\nabla^{\tilde m_1+1}u|\cdots\nabla^{\tilde m_i+1}u|\left[|\nabla^{k_1}u_t||\nabla^{k_2}u_t|\cdots\right], \tag{6.5}$$

where $\left[|\nabla^{k_1}u_t||\nabla^{k_2}u_t|\cdots\right]$ is as in the nonlinearity $\mathcal{N}(u)$ and $m_1\dots,m_j,\tilde m_1,\dots,\tilde m_i,k_1,k_2\dots$ are as used before. Hence by Lemma 2.2 for

$$f_1=w,\ f_2=\nabla h_1,\ \dots,f_j=\nabla h_{j-1},f_{j+1}=\nabla u,\dots,f_{i+j}=\nabla u,$$

and $f_{i+j+1},\ f_{i+j+2},\ (f_{i+j+3},\ f_{i+j+4})$, according to the different terms in $\mathcal{N}(u)$ as above, we estimate (6.5) in L^2 by (note $l_1>0,\ j\geq 1$ and $i+j<k-2$)

$$\begin{aligned}&\left(\|w\|_{L^\infty}^{1-\frac{m_1}{k-2-i-j}}\|w\|_{H^{k-2-i-j}}^{\frac{m_1}{k-2-i-j}}+\|w\|_{L^\infty}^{1-\frac{m_1}{k-1-i-j}}\|w\|_{H^{k-1-i-j}}^{\frac{m_1}{k-1-i-j}}\right)\\ &\cdot(1+\|\nabla u\|_{H^{k-1}}^{2k}+\|u_t\|_{H^{k-2}}^{2k}+\|\nabla v\|_{H^{k-1}}^{2k}+\|v_t\|_{H^{k-2}}^{2k})\\ &\lesssim\|w\|_{H^{k-1}}(1+\|\nabla u\|_{H^{k-1}}^{2k}+\|u_t\|_{H^{k-2}}^{2k}+\|\nabla v\|_{H^{k-1}}^{2k}+\|v_t\|_{H^{k-2}}^{2k}).\end{aligned}$$

We now continue with

$$\begin{aligned}&\nabla^{k-3}[(I-P_v)(\mathcal{N}(u)-\mathcal{N}(v))]\cdot\nabla^{k-3}w_t\\ &=\nabla^{k-3}(\mathcal{N}(u)-\mathcal{N}(v))(I-P_v)\nabla^{k-3}w_t+\sum_{\substack{l_1+l_2=k-3\\ l_1>0}}\nabla^{l_1}(I-P_v)\nabla^{l_2}(\mathcal{N}(u)-\mathcal{N}(v))\cdot\nabla^{k-3}w_t\\ &=\nabla^{k-3}(\mathcal{N}(u)-\mathcal{N}(v))\nabla^{k-3}[(P_u-P_v)u_t]-\sum_{\substack{l_1+l_2=k-3\\ l_1>0}}\nabla^{k-3}(\mathcal{N}(u)-\mathcal{N}(v))\cdot\nabla^{l_1}[(I-P_v)]\nabla^{l_2}w_t\\ &\quad+\sum_{\substack{l_1+l_2=k-3\\ l_1>0}}\nabla^{l_1}(I-P_v)\nabla^{l_2}(\mathcal{N}(u)-\mathcal{N}(v))\cdot\nabla^{k-3}w_t =: J_1+J_2+J_3.\end{aligned}$$

where the last equality follows from

$$(I-P_v)w_t=(I-P_v)u_t=[(I-P_v)-(I-P_u)]u_t=(P_u-P_v)u_t.$$

We use integration by parts to treat $\int J_1\ dx,\ \int J_2\ dx$. Therefore we assume $k\geq 4$, otherwise (if $k=3$) the estimate becomes easier and we only use integration by parts for $dP_v(\nabla^3 w\star\nabla u)$

in the difference $\mathcal{N}(u) - \mathcal{N}(v)$. We have

(6.6)
$$\int_{\mathbb{R}^n} J_1 \; dx = - \int_{\mathbb{R}^n} \nabla^{k-4}[\mathcal{N}(u) - \mathcal{N}(v)] \cdot \nabla^{k-2}[(P_u - P_v)u_t] \; dx, \quad \text{and}$$
(6.7)
$$\int_{\mathbb{R}^n} J_2 \; dx = \sum_{\substack{l_1+l_2=k-3\\ l_1>0}} \int_{\mathbb{R}^n} \nabla^{k-4}[\mathcal{N}(u) - \mathcal{N}(v)] \cdot [\nabla^{l_1+1}(I-P_v)\nabla^{l_2} w_t + \nabla^{l_1}(I-P_v)\nabla^{l_2+1} w_t] \; dx.$$

Thus we infer

$$\int_{\mathbb{R}^n} J_1 \; dx \le \left\|\nabla^{k-4}[\mathcal{N}(u) - \mathcal{N}(v)]\right\|_{L^2} \left\|\nabla^{k-2}[(P_u - P_v)u_t]\right\|_{L^2}$$

and from Corollary A.3, Lemma A.2 and Lemma 2.2

$$\left\|\nabla^{k-4}[\mathcal{N}(u) - \mathcal{N}(v)]\right\|_{L^2} \lesssim (\|w\|_{H^{k-1}} + \|w_t\|_{H^{k-3}})(1 + \|\nabla u\|_{H^{k-1}}^{2k} + \|u_t\|_{H^{k-2}}^{2k} + \|\nabla v\|_{H^{k-1}}^{2k} + \|v_t\|_{H^{k-2}}^{2k}),$$
$$\left\|\nabla^{k-2}[(P_u - P_v)u_t]\right\|_{L^2} \lesssim \|w\|_{H^{k-1}} (1 + \|\nabla u\|_{H^{k-1}}^{2k} + \|u_t\|_{H^{k-2}}^{2k} + \|\nabla v\|_{H^{k-1}}^{2k} + \|v_t\|_{H^{k-2}}^{2k}).$$

Similarly

$$\int_{\mathbb{R}^n} J_2 \; dx \le \sum_{\substack{l_1+l_2=k-3\\ l_1>0}} \left\|\nabla^{k-4}[\mathcal{N}(u) - \mathcal{N}(v)]\right\|_{L^2} (\left\|\nabla^{l_1+1}(I-P_v)\nabla^{l_2} w_t\right\|_{L^2} + \left\|\nabla^{l_1}(I-P_v)\nabla^{l_2+1} w_t\right\|_{L^2}),$$

for which we obtain similar upper bounds. For J_3, we note that

$$\int_{\mathbb{R}^n} J_3 \; dx \le \sum_{\substack{l_1+l_2=k-3\\ l_1>0}} \left\|\nabla^{l_1}(I-P_v)\nabla^{l_2}[\mathcal{N}(u) - \mathcal{N}(v)]\right\|_{L^2} \left\|\nabla^{k-3} w_t\right\|_{L^2},$$

and (note here $l_2 < k-3$) again by Corollary A.3

$$\begin{aligned}&\left\|\nabla^{l_1}(I-P_v)\nabla^{l_2}[\mathcal{N}(u) - \mathcal{N}(v)]\right\|_{L^2}\\ &\quad \lesssim (\|w\|_{H^{k-1}} + \|w_t\|_{H^{k-3}})(1 + \|\nabla u\|_{H^{k-1}}^{2k} + \|u_t\|_{H^{k-2}}^{2k} + \|\nabla v\|_{H^{k-1}}^{2k} + \|v_t\|_{H^{k-2}}^{2k}).\end{aligned}$$

Summing up, we have

$$\frac{d}{dt}\left(\int_{\mathbb{R}^n} |\nabla^{k-3} w_t|^2 + |\nabla^{k-1} w|^2 \; dx\right) \lesssim \mathcal{E}^2(t)(1 + \|\nabla u\|_{H^{k-1}}^{2k} + \|u_t\|_{H^{k-2}}^{2k} + \|\nabla v\|_{H^{k-1}}^{2k} + \|v_t\|_{H^{k-2}}^{2k}).$$

Note that we also obtain the following estimate (by integrating $dP_v(\nabla^3 w \star \nabla u)$ by parts) in a similar way

$$\frac{d}{dt}\left(\int_{\mathbb{R}^n} |w_t|^2 + |\Delta w|^2 \; dx\right) \lesssim \mathcal{E}^2(t)(1 + \|\nabla u\|_{H^{k-1}}^{2k} + \|u_t\|_{H^{k-2}}^{2k} + \|\nabla v\|_{H^{k-1}}^{2k} + \|v_t\|_{H^{k-2}}^{2k}).$$

Combining this, we use interpolation on the left-hand side in order to conclude

$$\frac{d}{dt}\mathcal{E}^2(t) \lesssim \mathcal{E}^2(t)(1 + \|\nabla u\|_{H^{k-1}}^{2k} + \|u_t\|_{H^{k-2}}^{2k} + \|\nabla v\|_{H^{k-1}}^{2k} + \|v_t\|_{H^{k-2}}^{2k}).$$

Since by assumption, for any $t \in (0, T)$, we have

$$\sup_{s\in[0,t)} (\|\nabla u\|_{H^{k-1}}^{2k} + \|u_t\|_{H^{k-2}}^{2k} + \|\nabla v\|_{H^{k-1}}^{2k} + \|v_t\|_{H^{k-2}}^{2k}) < \infty,$$

and $w(0) = 0$, we conclude the Lemma. $\square$

Appendix A. Derivatives of the nonlinearity

In this section we assume $u, v : \mathbb{R}^n \times [0,T) \to \mathbb{R}^L$ are smooth maps. The calculations hold if u and v are sufficiently regular to apply the Leibniz formula (e.g. with weak derivatives in L^2). Lemma 2.1 and the Leibniz formula imply the following

Lemma A.1. *Let $l \in \mathbb{N}$, then*

$$\nabla^l(\mathcal{N}(u)) = J_1 + J_2 + J_3,$$

where we $J_1,\ J_2,\ J_3$ are of the form (with $k_i,\ m_i \geq 0$)

$$J_1 = \sum_{(*)} \left(d^{j+1}P_u(\nabla^{m_1+1}u \star \dots \star \nabla^{m_j+1}u)[\nabla^{k_1}u_t \star \nabla^{k_2}u_t + \nabla^{k_1+2}u \star \nabla^{k_2+2}u + \nabla^{k_1+3}u \star \nabla^{k_2+1}u] \right),$$

with $(*) : 0 \leq m \leq l,\ \sum_{i=1}^2 k_i = l - m,\ j = \min\{1,m\}, \dots, m,\ \sum_k^j m_k = m - j.$

$$J_2 = \sum_{(*)} \left(d^{j+2}P_u(\nabla^{m_1+1}u \star \dots \star \nabla^{m_j+1}u)[\nabla^{k_1+1}u \star \nabla^{k_2+1}u \star \nabla^{k_3+2}u] \right),$$

with $(*) : 0 \leq m \leq l,\ \sum_{i=1}^3 k_i = l - m,\ j = \min\{1,m\}, \dots, m,\ \sum_k^j m_k = m - j.$

$$J_3 = \sum_{(*)} \left(d^{j+3}P_u(\nabla^{m_1+1}u \star \dots \star \nabla^{m_j+1}u)[\nabla^{k_1+1}u \star \nabla^{k_2+1}u \star \nabla^{k_3+1}u \star \nabla^{k_4+1}u] \right),$$

with $(*) : 0 \leq m \leq l,\ \sum_{i=1}^4 k_i = l - m,\ j = \min\{1,m\}, \dots, m,\ \sum_k^j m_k = m - j.$

The following Lemmata are used to prove the existence of a fixed point in Section 3 and the uniqueness result in Section 6.

Lemma A.2. *For $m \in \mathbb{N},\ k \in \mathbb{N}_0,$ we have for $m \geq 2$ and $w = u - v$*

(A.1)

$$\begin{aligned}\nabla^m(d^kP_u - d^kP_v) = &\sum_{j=1}^m \sum_{m_1+\dots m_j = m-j} (d^{j+k}P_u - d^{j+k}P_v)(\nabla^{m_1+1}u, \dots, \nabla^{m_j+1}u) \\ &+ \sum_{j=2}^m \sum_{m_1+\dots m_j = m-j} \sum_{h_1,\dots,h_{j-1} \in \{u,v\}} d^{j+k}P_v(\nabla^{m_1+1}w, \nabla^{m_2+1}h_1, \dots, \nabla^{m_j+1}h_{j-1}),\end{aligned}$$

and for $m = 1$

$$\nabla(d^kP_u - d^kP_v) = (dP_u - dP_v)(\nabla u) + dP_v(\nabla w). \tag{A.2}$$

Proof. This follows from subtracting the expansion in Lemma 2.1 for d^kP_v

$$\nabla^m(d^kP_v) = \sum_{j=1}^m \sum_{m_1+\dots m_j = m-j} d^{j+k}P_v(\nabla^{m_1+1}v, \dots, \nabla^{m_j+1}v),$$

from the same expansion of $\nabla^m(d^kP_u)$. Then subsequent adding and subtracting the intermediate terms in the formula above gives the result. $\square$

Corollary A.3. For $m \in \mathbb{N},\ m \geq 2,\ \ w = u - v$

$$\begin{aligned}&\nabla^m \left[(dP_u - dP_v)(u_t \cdot u_t + \nabla^2u \star \nabla^2u + \nabla^3u \star \nabla u) \right] \\ &\quad = \sum_{(*)} (d^{j+1}P_u - d^{j+1}P_v)(\nabla^{m_1+1}u, \dots, \nabla^{m_j+1}u)(\nabla^{k_1}u_t \star \nabla^{k_2}u_t + \nabla^{k_1+2}u \star \nabla^{k_2+2}u + \nabla^{k_1+3}u \star \nabla^{k_2}u) \\ &\qquad + \sum_{(**)} d^{j+1}P_v(\nabla^{m_1+1}w, \nabla^{m_2+1}h_1, \dots, \nabla^{m_j+1}h_{j-1})(\nabla^{k_1}u_t \star \nabla^{k_2}u_t + \nabla^{k_1+2}u \star \nabla^{k_2+2}u + \nabla^{k_1+3}u \star \nabla^{k_2}u),\end{aligned}$$

$$\begin{aligned}
&\nabla^m \left[(d^2P_u - d^2P_v)(\nabla u \star \nabla u \star \nabla^2 u)\right] \\
&\quad = \sum_{(*)} (d^{j+2}P_u - d^{j+2}P_v)(\nabla^{m_1+1}u, \dots, \nabla^{m_j+1}u)(\nabla^{k_1+1}u \star \nabla^{k_2+1}u \star \nabla^{k_3+2}u) \\
&\qquad + \sum_{(**)} d^{j+2}P_v(\nabla^{m_1+1}w, \nabla^{m_2+1}h_1, \dots, \nabla^{m_j+1}h_{j-1})(\nabla^{k_1+1}u \star \nabla^{k_2+1}u \star \nabla^{k_3+2}u),
\end{aligned}$$

$$\begin{aligned}
&\nabla^m \left[(d^3P_u - d^3P_v)(\nabla u \star \nabla u \star \nabla u \star \nabla u)\right] \\
&\quad = \sum_{(*)} (d^{j+3}P_u - d^{j+3}P_v)(\nabla^{m_1+1}u, \dots, \nabla^{m_j+1}u)(\nabla^{k_1+1}u \star \nabla^{k_2+1}u \star \nabla^{k_3+1}u \star \nabla^{k_4+1}u) \\
&\qquad + \sum_{(**)} d^{j+3}P_v(\nabla^{m_1+1}w, \nabla^{m_2+1}h_1, \dots, \nabla^{m_j+1}h_{j-1})(\nabla^{k_1+1}u \star \nabla^{k_2+1}u \star \nabla^{k_3+1}u \star \nabla^{k_4+1}u),
\end{aligned}$$

where

$(*) : j = 1, \dots, m$ and $m_1 + \dots + m_j + k_1 + k_2 = m - j,\ m_1 + \dots + m_j + k_1 + k_2 + k_3 = m - j,$
$m_1 + \dots + m_j + k_1 + k_2 + k_3 + k_4 = m - j,$ respectively and,
$(**) : j = 2, \dots, m$ and $m_1 + \dots + m_j + k_1 + k_2 = m - j,\ m_1 + \dots + m_j + k_1 + k_2 + k_3 = m - j,$
$m_1 + \dots + m_j + k_1 + k_2 + k_3 + k_4 = m - j,\ h_1, \dots, h_{j-1} \in \{u, v\}.$

Also, the case $m = 1$ is similar.

Proof. This follows again from the Leibniz rule and the application of Lemma A.2. □

Corollary A.4. We have for $m \in \mathbb{N},\ m \geq 2$ and $w = u - v$ that

$$\nabla^m(\mathcal{N}(u) - \mathcal{N}(v))$$

is a linear combination of the terms

$$(d^{j+1}P_u - d^{j+1}P_v)(\nabla^{m_1+1}u, \dots, \nabla^{m_j+1}u)(\nabla^{k_1}u_t \star \nabla^{k_2}u_t + \nabla^{k_1+2}u \star \nabla^{k_2+2}u + \nabla^{k_1+3}u \star \nabla^{k_2}u),$$
$$d^{j+1}P_v(\nabla^{m_1+1}w, \nabla^{m_2+1}h_1, \dots, \nabla^{m_j+1}h_{j-1})(\nabla^{k_1}u_t \star \nabla^{k_2}u_t + \nabla^{k_1+2}u \star \nabla^{k_2+2}u + \nabla^{k_1+3}u \star \nabla^{k_2}u),$$

$$(d^{j+2}P_u - d^{j+2}P_v)(\nabla^{m_1+1}u, \dots, \nabla^{m_j+1}u)(\nabla^{k_1+1}u \star \nabla^{k_2+1}u \star \nabla^{k_3+2}u),$$
$$d^{j+2}P_v(\nabla^{m_1+1}w, \nabla^{m_2+1}h_1, \dots, \nabla^{m_j+1}h_{j-1})(\nabla^{k_1+1}u \star \nabla^{k_2+1}u \star \nabla^{k_3+2}u),$$

$$(d^{j+3}P_u - d^{j+3}P_v)(\nabla^{m_1+1}u, \dots, \nabla^{m_j+1}u)(\nabla^{k_1+1}u \star \nabla^{k_2+1}u \star \nabla^{k_3+1}u \star \nabla^{k_4+1}u),$$
$$d^{j+3}P_v(\nabla^{m_1+1}w, \nabla^{m_2+1}h_1, \dots, \nabla^{m_j+1}h_{j-1})(\nabla^{k_1+1}u \star \nabla^{k_2+1}u \star \nabla^{k_3+1}u \star \nabla^{k_4+1}u), \quad \text{and}$$

$$\begin{aligned}
&d^{j+1}P_v(\nabla^{m_1+1}v, \dots, \nabla^{m_j+1}v)(\nabla^{k_1}w_t \star \nabla^{k_2}h_t + \nabla^{k_1+2}w \star \nabla^{k_2+2}h \\
&\quad + \nabla^{k_1+3}w \star \nabla^{k_2}h + \nabla^{k_1+3}h \star \nabla^{k_2}w), \quad h \in \{u, v\},
\end{aligned}$$

$$d^{j+2}P_v(\nabla^{m_1+1}v, \dots, \nabla^{m_j+1}v)(\nabla^{k_1+1}w \star \nabla^{k_2+1}h_1 \star \nabla^{k_3+2}h_2 + \nabla^{k_1+1}h_1 \star \nabla^{k_2+1}h_2 \star \nabla^{k_3+2}w),$$

$$d^{j+3}P_v(\nabla^{m_1+1}v, \dots, \nabla^{m_j+1}v)(\nabla^{k_1+1}w \star \nabla^{k_2+1}h_1 \star \nabla^{k_3+1}h_2 \star \nabla^{k_4+1}h_3),$$

where j, k_1, k_2, k_3, k_4 and h_1, h_2, h_3, h_4 are as above in Corollary A.3. Also, we have a similar (but simpler) statement for $m = 1$.

Proof. We write, according to the definition of $\mathcal{N}(u)$ in (1.3),

$$\begin{aligned}\mathcal{N}(u)-\mathcal{N}(v) &= (dP_u-dP_v)(u_t\cdot u_t+\nabla^2u\star\nabla^2u+\nabla^3u\star\nabla u)\\ &\quad+(d^2P_u-d^2P_v)(\nabla u\star\nabla u\star\nabla^2u)+(d^3P_u-d^3P_v)(\nabla u\star\nabla u\star\nabla u\star\nabla u)\\ &\quad+dP_v(w_t\cdot u_t+v_t\cdot w_t+\nabla w\star\nabla u+\nabla v\star\nabla w+\nabla^3w\star\nabla u+\nabla^3v\star\nabla w)\\ &\quad+d^2P_v(\nabla w\star\nabla u\star\nabla^2u+\nabla v\star\nabla w\star\nabla^2u+\nabla v\star\nabla v\star\nabla^2w)\\ &\quad+d^3P_v(\nabla w\star\nabla u\star\nabla u\star\nabla u+\nabla v\star\nabla w\star\nabla u\star\nabla u\\ &\qquad+\nabla v\star\nabla v\star\nabla w\star\nabla u+\nabla v\star\nabla v\star\nabla v\star\nabla w).\end{aligned}$$

Then, we use Corollary A.3 for the first three terms in the sum above. For the latter three, we use Lemma 2.1 and the Leibniz rule. □

We recall that in Section 3 for $\varepsilon\in(0,1)$ by definition

$$\mathcal{N}(u)_\varepsilon=\mathcal{N}(u)-\varepsilon d^2P_u(u_t,\nabla u,\nabla u)-\varepsilon 2dP_u(\nabla u_t,\nabla u)-\varepsilon dP_u(u_t,\Delta u).$$

Lemma A.5. *For $m\in\mathbb{N}_0$ the derivative $\nabla^m(\mathcal{N}_\varepsilon(u))$ consists of the following additional terms*

$$d^{j+1}P_u(\nabla^{m_1+1}u\star\cdots\star\nabla^{m_j+1}u)(\nabla^{k_1}u_t\star\nabla^{k_2+2}u+\nabla^{k_1+1}u_t\star\nabla^{k_2+1}u),\quad \textit{and}$$
$$d^{j+2}P_u(\nabla^{m_1+1}u\star\cdots\star\nabla^{m_j+1}u)(\nabla^{k_1}u_t\star\nabla^{k_2+1}u\star\nabla^{k_3+1}u),$$

with $j,m_1,\dots,m_j,k_1,k_2,k_3$ similarly to Lemma A.1.
Further $\nabla^m(\mathcal{N}_\varepsilon(u))-\nabla^m(\mathcal{N}_\varepsilon(v))$ consists of additional terms of the form

$$(d^{j+1}P_u-d^{j+1}P_v)(\nabla^{m_1+1}u,\dots,\nabla^{m_j+1}u)(\nabla^{k_1}u_t\star\nabla^{k_2+2}u+\nabla^{k_1+1}u_t\star\nabla^{k_2+1}u),$$
$$d^{j+1}P_v(\nabla^{m_1+1}w,\nabla^{m_2+1}h_1,\dots,\nabla^{m_j+1}h_{j-1})(\nabla^{k_1}u_t\star\nabla^{k_2+2}u+\nabla^{k_1+1}u_t\star\nabla^{k_2+1}u),$$

$$(d^{j+2}P_u-d^{j+2}P_v)(\nabla^{m_1+1}u,\dots,\nabla^{m_j+1}u)(\nabla^{k_1}u_t\star\nabla^{k_2+1}u\star\nabla^{k_3+1}u),$$
$$d^{j+2}P_v(\nabla^{m_1+1}w,\nabla^{m_2+1}h_1,\dots,\nabla^{m_j+1}h_{j-1})(\nabla^{k_1}u_t\star\nabla^{k_2+1}u\star\nabla^{k_3+1}u),\quad \textit{and}$$

$$\begin{aligned}&d^{j+1}P_v(\nabla^{m_1+1}v,\dots,\nabla^{m_j+1}v)(\nabla^{k_1}w_t\star\nabla^{k_2+2}h+\nabla^{k_1+1}w_t\star\nabla^{k_2+1}h\\ &\quad+\nabla^{k_1}h\star\nabla^{k_2+2}w+\nabla^{k_1+1}h_t\star\nabla^{k_2+1}w),\quad h\in\{u,v\},\end{aligned}$$

$$d^{j+2}P_v(\nabla^{m_1+1}v,\dots,\nabla^{m_j+1}v)(\nabla^{k_1}w_t\star\nabla^{k_2+1}h_1\star\nabla^{k_3+1}h_2+\nabla^{k_1}(h_1)_t\star\nabla^{k_2+1}h_2\star\nabla^{k_3+1}w),$$

where $w=u-v$ and $j,m_1,\dots,m_j,k_1,k_2,k_3,h_1,\dots,h_{j-1}$ similarly to Corollary A.4.

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